Note Title 11.02.2020

$$S^2 = \mathbb{C} \,\dot\cup\, \mathbb{C} \big/ z \sim \phi(z) = \frac{1}{z}, \quad z \neq 0$$

$$T_* S^2 = T_*\mathbb{C} \,\dot\cup\, T_*\mathbb{C} \big/ (z, w) \sim \Big(\phi(z), \overbrace{D\phi(z)}^{= -\frac{1}{z^2}}(w)\Big)$$

$$(z, w) \in T_*\mathbb{C}, \quad z \neq 0$$

$$= \mathbb{C} \times \mathbb{C} \,\dot\cup\, \mathbb{C} \times \mathbb{C} \big/ (z, w) \sim \left(\frac{1}{z}, -\frac{1}{z^2} \cdot w\right)$$

$$z \neq 0$$

$\underline{\underline{Ex}}$ $S^4 \doteq \mathbb{HP}^1 = \mathbb{H} \,\dot{\cup}\, \mathbb{H} / p \sim \frac{1}{p}, \; p \neq 0$

$\mathbb{H} = \mathbb{R}^4, \; p = (x_1, x_2, x_3, x_4) = x_1 + \hat{i}\, x_2 + \hat{j}\, x_3 + k x_4$

$\bar{p} = x_1 - \hat{i}\, x_2 - \hat{j}\, x_3 - k x_4$

$p\bar{p} = x_1^2 + x_2^2 + x_3^2 + x_4^2 \Rightarrow p \frac{\bar{p}}{\|p\|} = 1, \; p \neq 0.$

$p^{-1} = \frac{1}{p} = \frac{\bar{p}}{\|p\|}.$

$$T_* S^4 = T_* \mathbb{H} \,\dot{\cup}\, T_* \mathbb{H} \Big/ (p, v) \sim \left(\frac{1}{p}, D\phi(p)(v)\right)$$

$$\phi : \mathbb{H}^* \to \mathbb{H}^*, \quad \phi(p) = \frac{1}{p}, \quad p \in \mathbb{H}^*.$$

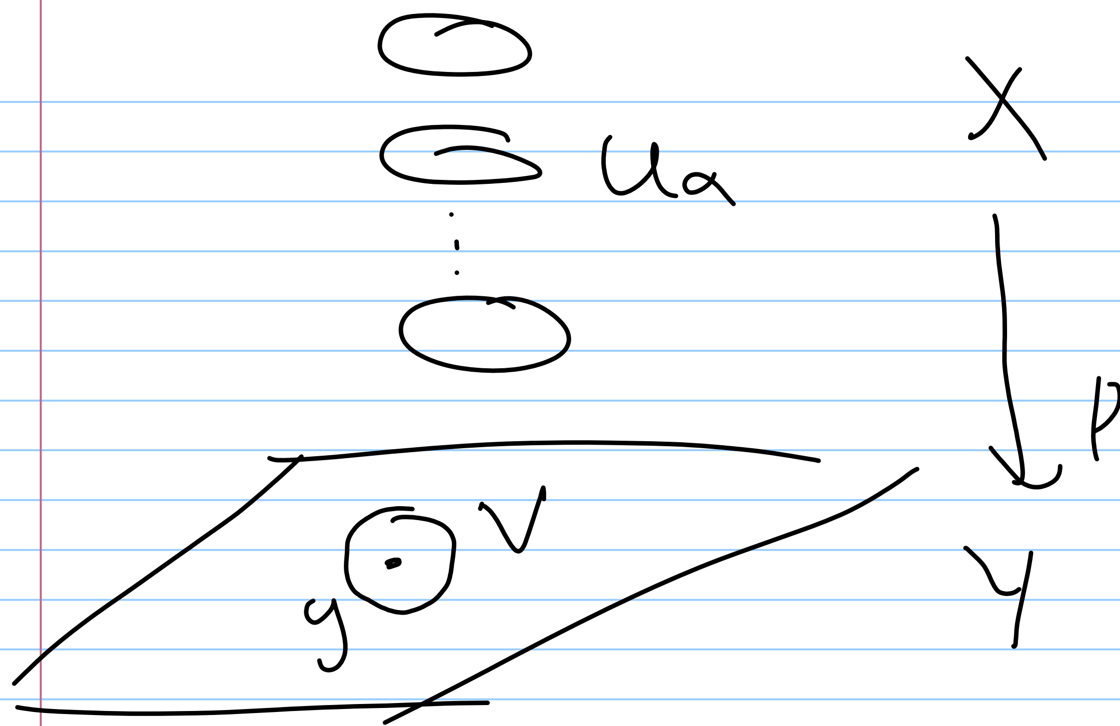
$$D\phi(p)(v) = -\frac{1}{p}\, v \,\frac{1}{p}$$

Qutient Manifolds:

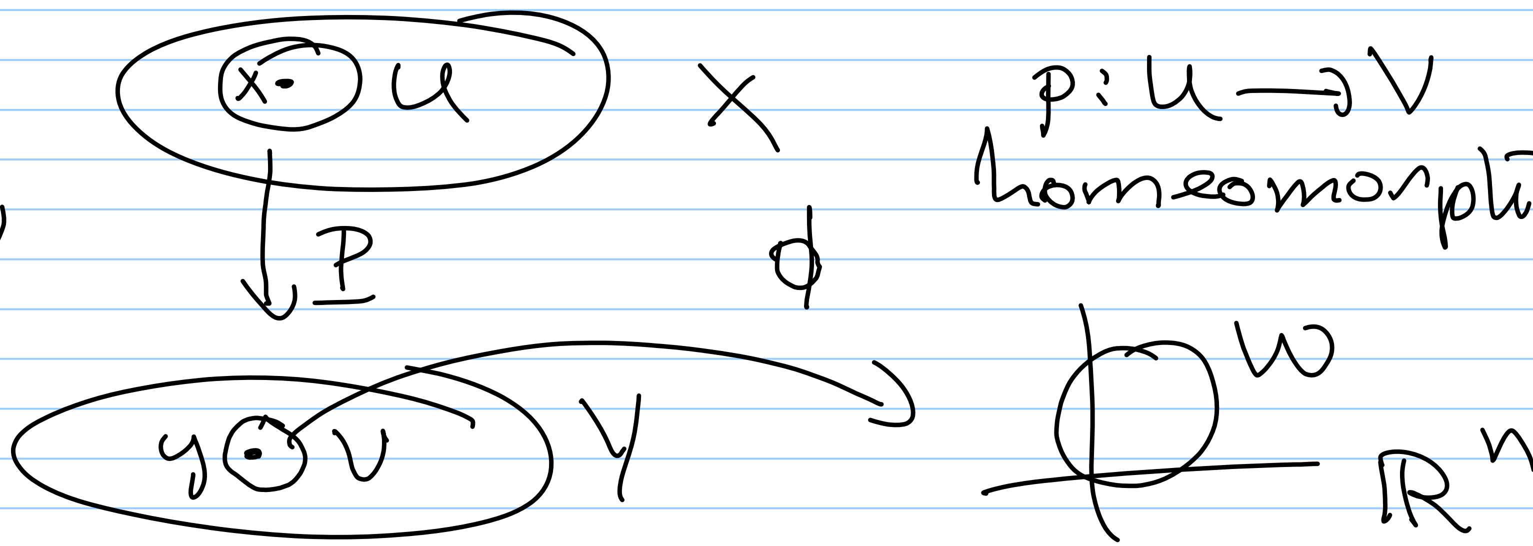
X, Y topological spaces, $p: X \to Y$ cont. map. p is called a covering space if for any $y \in Y$ there is a neighborhood V of y with

i) $y \in V$

ii) $p^{-1}(V)$ is a disjoint union open subsets $\{U_\alpha\}$ of X, when for each α the restriction map $p: U_\alpha \to V$ is a homeomorphism.


U_α
X
p
Y
y
V

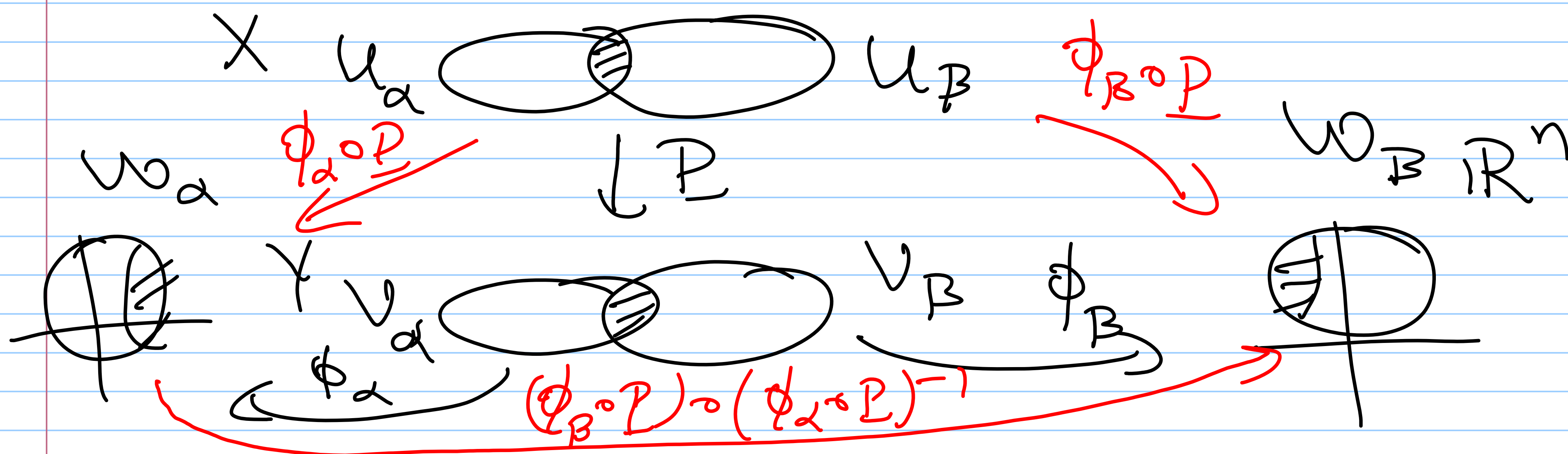
If Y has a smooth structure then X gets a smooth structure as follows:



$p : U \to V$ homeomorphism

$y = p(x)$

The $\phi \circ P : U \longrightarrow W$ is a coordinate system about $x \in X$.

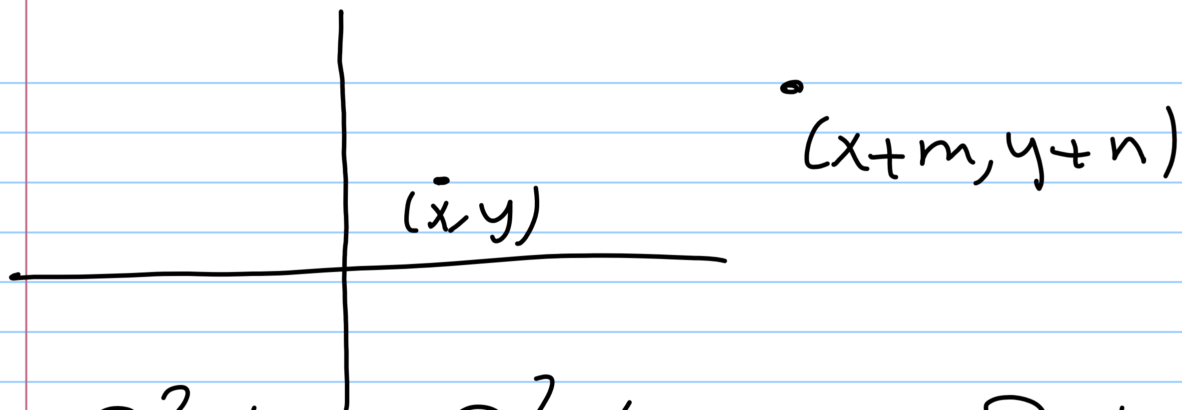


$$(\phi_\beta \circ \Phi) \circ (\phi_\alpha \circ \Phi)^{-1} = \phi_\beta \circ \underbrace{(\Phi \circ \Phi^{-1})}_{= \mathrm{Id}} \circ \phi_\alpha^{-1}$$

$$= \phi_\beta \circ \phi_\alpha^{-1} \in C^\infty,$$ because Y has smooth structure.

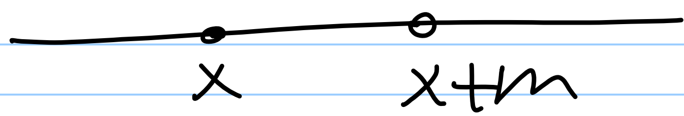
<u>Ex</u>: $G = \mathbb{Z} \times \mathbb{Z}$, $G \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

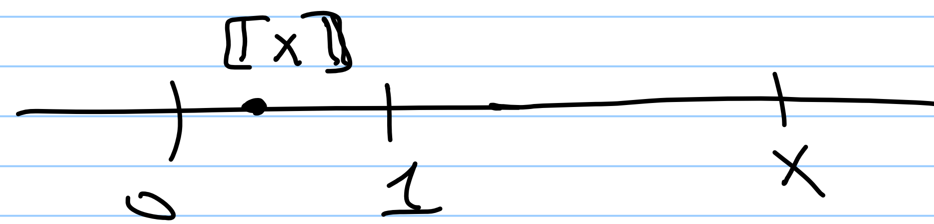
$$(m,n) \cdot (x,y) = (x+m, y+n)$$



$(m,n) \in G = \mathbb{Z} \times \mathbb{Z}$

$$\mathbb{R}^2/G = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = S^1 \times S^1$$





$$\mathbb{R}/\mathbb{Z} \simeq [0,1]/0\sim 1 = S^1$$

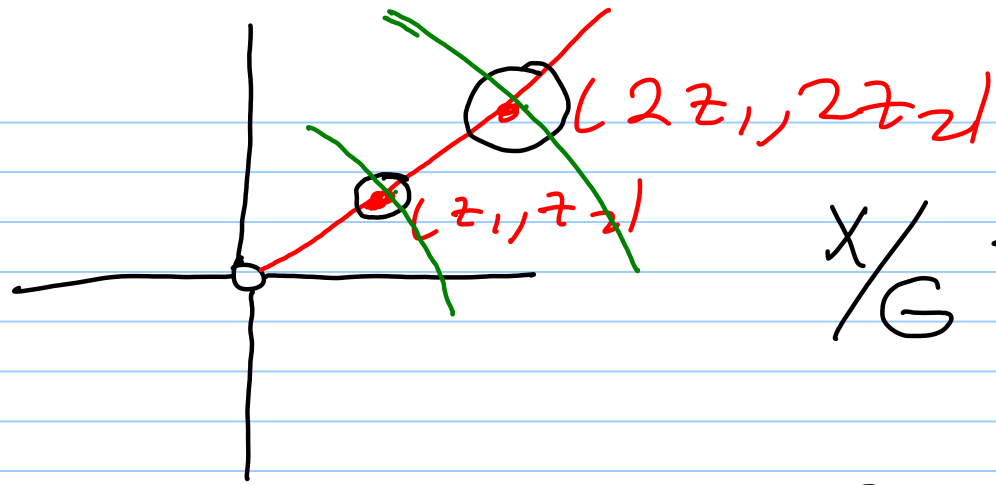
$h : \mathbb{R} \longrightarrow S^1$, $h(t) = (\cos 2\pi t, \sin 2\pi t)$

$h(t) = h(t+m)$ and

$h(t_1) = h(t_2)$ implies $t_1 - t_2 \in \mathbb{Z}$.

2) $G = \mathbb{Z}$, $X = \mathbb{C}^2 \setminus \{(0,0)\}$

$G \times X \longrightarrow X$, $n \cdot (z_1, z_2) = (2^n z_1, 2^n z_2)$



$$X/G \cong S^3 \times S^1$$

$$\mathbb{C}^2 \setminus \{(0,0)\} / \mathbb{Z} \longrightarrow S^3 \times S^1$$

$$(z_1, z_2) \longmapsto \left(\frac{(z_1, z_2)}{\|(z_1, z_2)\|},\ e^{2\pi i \log_2 \|(z_1, z_2)\|} \right)$$

diffeomorphism.

Rank Theorems:

Definition: Let $f: M \to N$ be a smooth map of smooth manifolds and $p \in M$. If $Df(p): T_pM \to T_{f(p)}N$ is injective then we say that f is an immersion at p. If $Df(p): T_pM \to T_{f(p)}N$ is onto then we say that f is an submersion at p.

Ex: $m \leq n$, $f: \mathbb{R}^m \to \mathbb{R}^n$,

$$(x_1, \ldots, x_m) \longmapsto (x_1, \ldots, x_m, \overbrace{0, \ldots, 0}^{n-m})$$

$Df(p): T_p\mathbb{R}^m \longrightarrow \mathbb{R}^n$, $\quad v = (v_1, \ldots, v_m) \in T_p\mathbb{R}^n$

$Df(p)(v) = (v_1, \ldots, v_m, 0, \ldots, 0)$ is clearly 1-1.

This is called the canonical immersion.

If $m \geq n$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $g(x_1, \ldots, x_n, \ldots, x_m) = (x_1, \ldots, x_n)$.
Clearly, $Dg(p) : T_p\mathbb{R}^m \to T_{g(p)}\mathbb{R}^n$ is given by
$Dg(p)(v_1, \ldots, v_n, \ldots, v_m) = (v_1, \ldots, v_n)$. So $Dg(p)$
is clearly onto and thus g is an
submersion, called the canonical submersion.

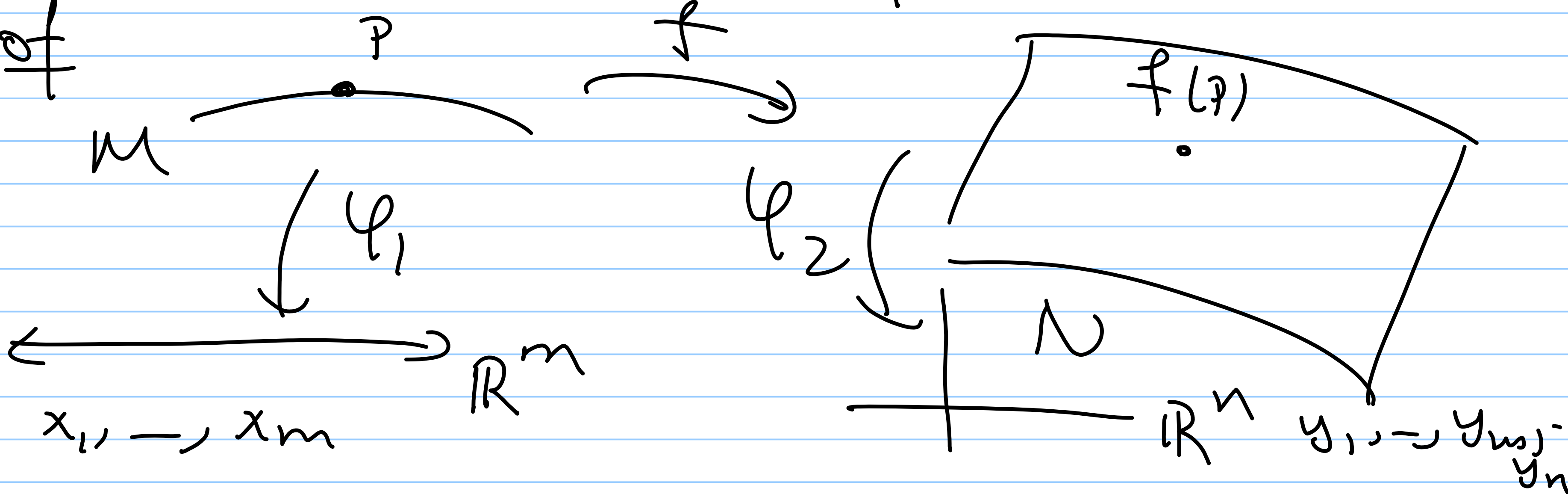
__Theorem:__ Let $f: M \to N$ be a smooth map and $p \in M$ so that f is an immersion at p. Then one can find coordinate charts around p and $f(p)$, say $\varphi_1 : U_1 \to V_1$, $\varphi_2 : U_2 \to V_2$, $p_1 \in U_1 \subseteq M$, $f(p) \in U_2 \subseteq N$, $V_1 \subseteq \mathbb{R}^m$, $V_2 \subseteq \mathbb{R}^n$, so that

$$\begin{tikzcd} U_1 \arrow[r, "f"] \arrow[d, "\varphi_1"'] & U_2 \arrow[d, "\varphi_2"] \\ \mathbb{R}^m \supseteq V_1 \arrow[r, "\varphi_2 \circ f \circ \varphi_1^{-1}"] & V_2 \subseteq \mathbb{R}^n \end{tikzcd}$$

$$(\varphi_2 \circ f \circ \varphi_1^{-1})(x_1, \dots, x_m) = (x_1, \dots, x_n, 0, \dots, 0).$$

Similar statement holds for submersions.

Proof



$Df(p) : T_pM \longrightarrow T_{f(p)}N$ is injective

$$Df(p) = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}}_{n \times m}$$

$f = (f_1, \ldots, f_n)$

$f_i = f_i(x_1, \ldots, x_m)$

Assume that the first m-rows of $Df(p)$ are linearly independent.

Let $g : \mathbb{R}^m \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$ be given by

$$g(x_1, \dots, x_m, x_{m+1}, \dots, x_n) := f(x_1, \dots, x_m) + (0, \dots, 0, x_{m+1}, \dots, x_n)$$

$$= (f_1, f_2, \dots, f_m, f_{m+1} + x_{m+1}, \dots, f_n + x_n)$$

$$Dg(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & 0 & 0 & \cdots & 0 \\ * & & * & 1 & 0 & \cdots & 0 \\ * & \cdots & * & 0 & 1 & \cdots & 0 \\ * & & * & & & \cdots & 1 \end{bmatrix}$$

Now Dg is invertible, $g : \underbrace{\mathbb{R}^m \times \mathbb{R}^{n-m}}_{\mathbb{R}^n} \longrightarrow \mathbb{R}^n$.

So by the Inverse function theorem g is a local diffeomorphism.

$$\begin{tikzcd}
p & & f(p) & \\
M \arrow[rr, "f"] \arrow[d, "\varphi_1"] & & N \arrow[d, "\varphi_2"] \arrow[drr, "g^{-1} \circ \varphi_2"] & & \\
\mathbb{R}^m \arrow[rr, "\varphi_2 \circ f \circ \varphi_1^{-1}"] & & \mathbb{R}^n \arrow[rr, "g^{-1}"] & & \mathbb{R}^n
\end{tikzcd}$$

$(x_1, \dots, x_m) \longmapsto (f_1, \dots, f_n) \longmapsto g^{-1}(f_1, \dots, f_n)$, where

$$\begin{tikzcd}[column sep=large] (x_1, \dots, x_m, x_{m+1}, \dots, x_n) \arrow[r, bend right, "g"'] & (f_1, \dots, f_m, f_{m+1} + x_{m+1}, \dots, f_n + x_n) \arrow[l, bend right=60, "g^{-1}"] \end{tikzcd}$$

<u>Claim:</u> $g^{-1}(f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)) = (x_1, \dots, x_m, 0, \dots, 0)$

Because, $g(x_1, \dots, x_m, 0, \dots, 0) = (f_1, \dots, f_m, f_{m+1}, \dots, f_n)$.