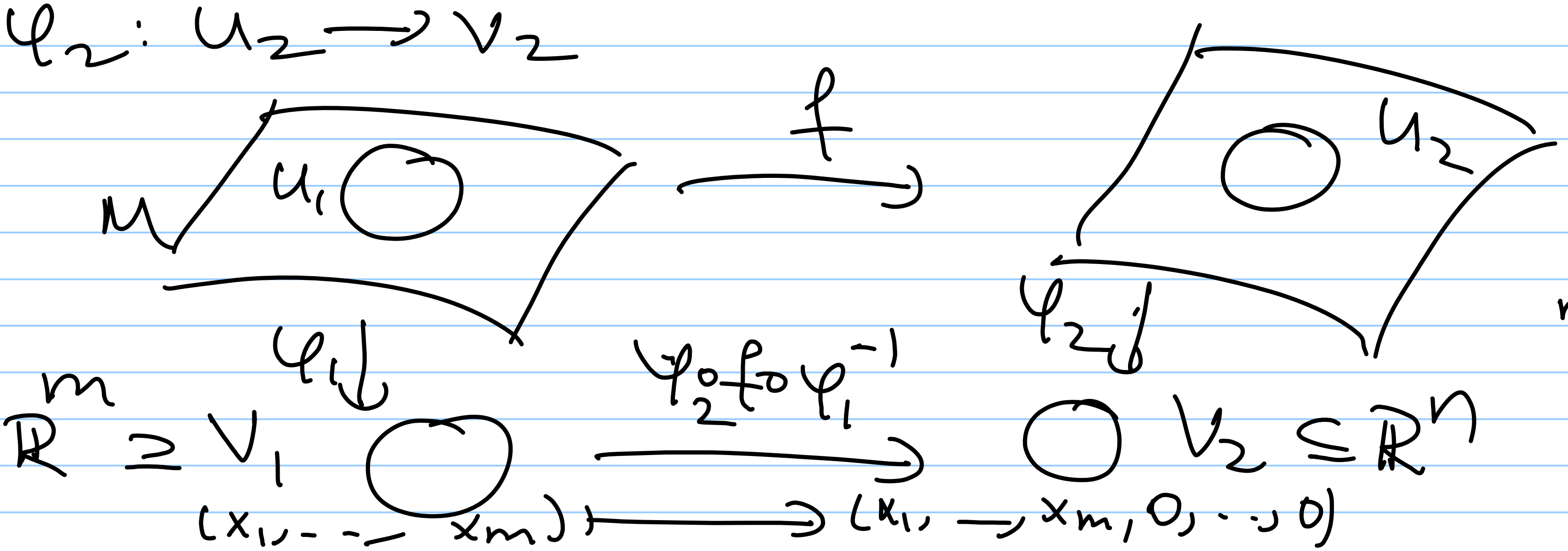
Note Title 12.02.2020

$f : M \longrightarrow N$ immersion at a point $p \in M$

$p \in U_1 \subseteq M$, $f(p) \in U_2 \subseteq N$, $\varphi_1 : U_1 \longrightarrow V_1$

$\varphi_2 : U_2 \longrightarrow V_2$



$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_m, 0, \dots, 0)$

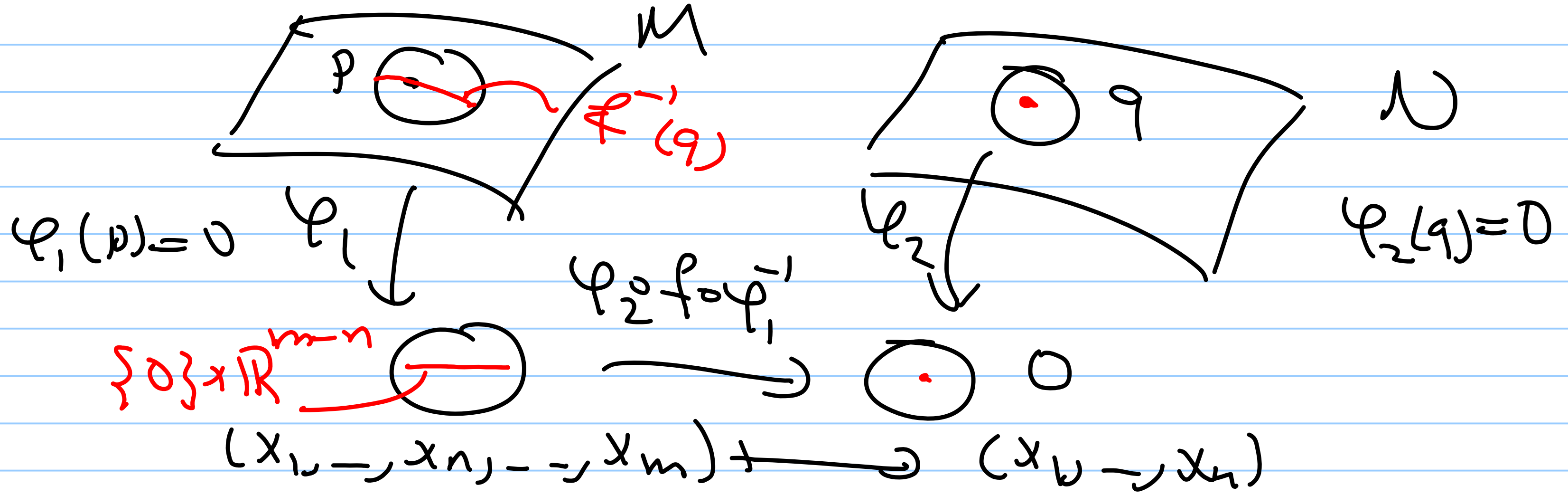
$m \leq n$

If f is a submersion then φ_i's can be chosen that $\varphi_2 \circ f \circ \varphi_1^{-1}$ becomes standard submersion.

$$\varphi_2 \circ f \circ \varphi_1^{-1}(x_1, \dots, x_n, \dots x_m) = (x_1, \dots, x_n) \qquad (m \geq n).$$

<u>Definition:</u> $f : M \longrightarrow N$ smooth map. A point $q \in N$ is called a regular value of $Df(p) : T_pM \longrightarrow T_qN$ is onto for all $p \in f^{-1}(q)$.

In this case, since $Df(p): T_pM \to T_qN$ is onto, f is an submersion at $p \in M$ and in some coordinate systems



So, $f^{-1}(q)$ becomes in this coordinate system

$$\varphi_1(f^{-1}(q)) = \{(x_1, \dots, x_n, \dots x_m) \in \mathbb{R}^m \mid x_1 = \dots = x_n = 0\}$$
$$= \{0\} \times \mathbb{R}^{m-n}$$

This implies that $f^{-1}(q)$ is an $m-n$-dimensional submanifold of M.

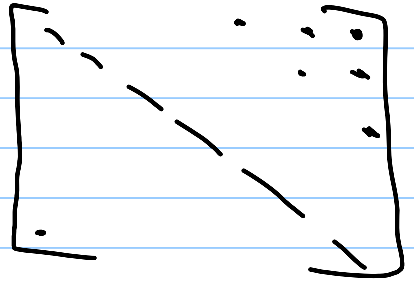
<u>Example:</u> $\Phi : M(n,n) \longrightarrow S(n)$, $\Phi(Q) = Q^T Q$

$M(n,n) = \mathbb{R}^{n \times n} = \mathbb{R}^{n^2}$: the set of all $n \times n$-real matrices

$S(n) = \{A \in M(n,n) \mid A^T = A\}$, the set of all

$n \times n$ - symmetric real matrices.

$$S(n) = \{ [a_{ij}]_{n \times n} \mid a_{ij} = a_{ji} \ \forall i,j \}$$
$$= \mathbb{R}^{\frac{n(n+1)}{2}}$$



$1+2+\cdots+n = \frac{n(n+1)}{2}$

$\Phi : M(n,n) \longrightarrow S(n)$, $Q \longmapsto Q^T Q$.

$$\begin{tikzcd} D\Phi(Id) : T_{Id} M(n,n) \arrow[r] \arrow[d, phantom, "\shortparallel"] & T_{Id} S(n) \arrow[d, phantom, "\shortparallel"] \\ M(n,n) & S(n) \\ A \arrow[r, mapsto] & A + A^T \end{tikzcd}$$

<u>$D\Phi$ is onto:</u> If $Q \in S(n)$, then $Q = \frac{Q}{2} + \frac{Q^T}{2} = D\Phi_{(Id)}\left(\frac{Q}{2}\right)$.

Hence, $Id \in S(n)$ is a regular value of Φ.

Therefore, $\Phi^{-1}(Id)$ is a smooth submanifold of $M(n,n)$ of dimension $\dim M(n,n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

Moreover,

$$\Phi^{-1}(Id) = \{ Q \in M(n,n) \mid Q^T Q = Id \}$$

$= O(n)$ the set of $n \times n$ orthogonal real matrices.

Sard's Theorem: $\Omega \subseteq \mathbb{R}^n$ open subset, $F: \Omega \to \mathbb{R}^m$ smooth map. let $C \subseteq \Omega$ be the set of all critical points of F. $C = \{ p \in \Omega \mid DF(p): T_p\Omega \to T_{F(p)}\mathbb{R}^m$ is not onto $\}$.

Then the $F(C)$ has measure zero in $\mathbb{R}^m$.

Ex: $n < m$, $DF(p): \underset{\mathbb{R}^n}{\underset{\shortparallel}{T_p\Omega}} \longrightarrow \underset{\mathbb{R}^m}{\underset{\shortparallel}{T_{F(p)}\mathbb{R}^m}}$ cannot be onto

Hence, the set of critical points C of F in Ω is Ω. So by Sard's Theorem $F(C)$, the set of critical values of F has measure zero. So the image $F(\Omega)$ has measure zero.

An application of Sard's Theorem: Embedding of smooth manifolds into Euclidean Spaces.

<u>Theorem:</u> Let M be a smooth n-dimensional manifold. Then there is an immersion of M into $\mathbb{R}^{2n}$ and an embedding into $\mathbb{R}^{2n+1}$.

<u>Idea:</u> First embedd M into $\mathbb{R}^N$ for some big N.

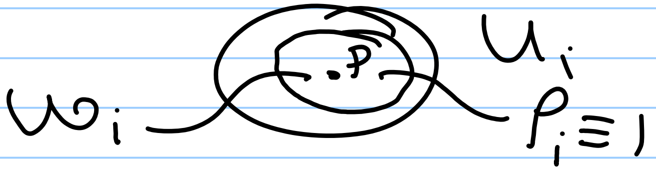
M compact $p \in M$, $p \in U \subseteq M$, $\varphi : U \to V \subseteq \mathbb{R}^n$

$M = U_1 \cup \cdots \cup U_k$, $\varphi_i : U_i \longrightarrow V_i \subseteq \mathbb{R}^n$

Extend φ_i to all M as a smooth function.

Then
$$\begin{array}{rcl} M & \longrightarrow & \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{nk} \\ p & \longmapsto & (\varphi_1(p), \varphi_2(p), \dots, \varphi_k(p)) \end{array}$$

To map this map also one to one we choose for each i a smooth funtion $\rho_i : M \to [0,1]$ so that it takes the value 1 in an open set in U_i containing p.



$W_i \subset \overline{W_i} \subset U_i$

$M = \cup W_i$

Then the map $F : M \longrightarrow \mathbb{R}^N$,

$F = (\varphi_1, -, \varphi_k, f_1, f_2, -, f_k)$ is a 1-1 immersion.

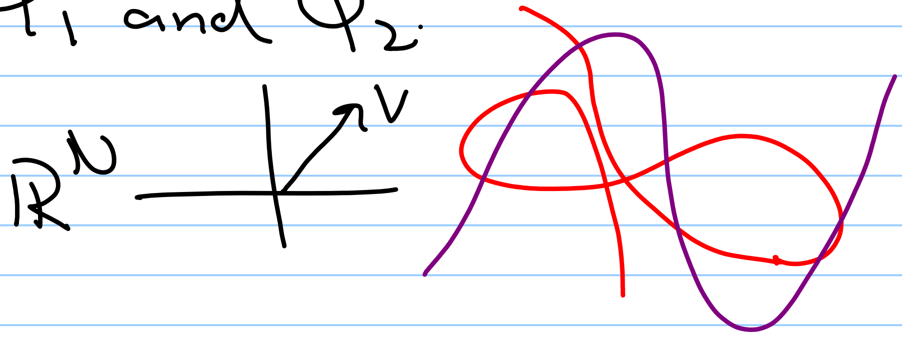
Moreover, M is compact and thus any 1-1 immersion is an embedding.

So we may assume M is a submanifold of some $\mathbb{R}^N$. Assume that $N > 2n+1$. Consider the functions

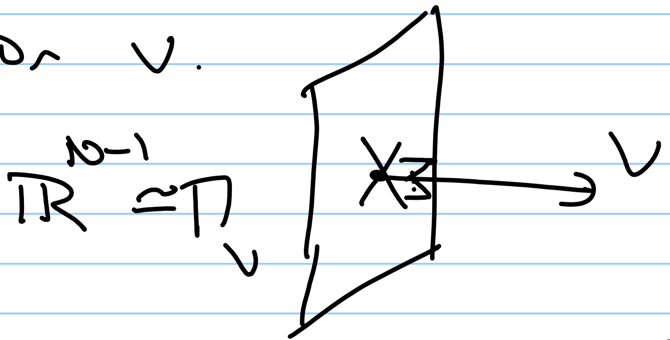
$\Psi_1 : M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^N$, $(x_1, x_2, \lambda) \longmapsto \lambda(x_1 - x_2)$ and

$\Psi_2 : T_x M \longrightarrow \mathbb{R}^N$, $(x, v) \longmapsto v$, $(x, v) \in T_xM \subseteq T_x\mathbb{R}^N \subseteq \mathbb{R}^N$

$N > 2n+1 > 2n$ and thus $\operatorname{Im}\Psi_1$ and $\operatorname{Im}\Psi_2$ consists of critical values of Ψ_1 and Ψ_2, respectively. In particular, Ψ_1 and Ψ_2 have measure zero. Since any ball in $\mathbb{R}^N$ has positive measure there is some vector $v \in \mathbb{R}^N$, $v \neq 0$, not contained in the images of Ψ_1 and Ψ_2.



Let Γ_v be the hypersurface in $\mathbb{R}^N$ perpendicular to the vector v.



Let $\pi : \mathbb{R}^N \longrightarrow \Gamma_v \simeq \mathbb{R}^{N-1}$ be the orthogonal projection.

<u>Claim:</u> The composition

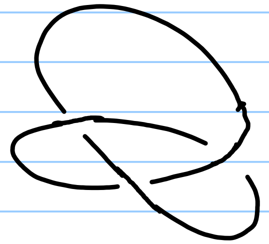
$$\pi \circ F : M \longrightarrow \Gamma_v \simeq \mathbb{R}^{N-1}$$

is still a one-to-one Immersion.

Proof Exercise.

This finishes the proof. $\equiv$

<u>Ex</u>



<u>Ex</u> $S^1 \times S^3 = \mathbb{C}^2 \setminus \{(0,0)\} \big/ (z_1, z_2) \sim (2z_1, 2z_2)$

$\cap$

$\mathbb{R}^2 \times \mathbb{R}^4 = \mathbb{R}^6$

$S^1 \times S^3 \not\subset \mathbb{C}^N$ as a complex submanifold. ($S^1 \times S^3$ compact)

$S^1 \times S^3 \not\subset \mathbb{CP}^N$ as a complex submanifold.
(Homology of $S^1 \times S^3$ is <u>not</u> suitable)

<u>Differential Forms:</u> $U \subseteq \mathbb{R}^n$ open subset

$p \in U$, T_pU tangent space: $T_pU = \text{span}\left\{ \frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_n}\Big|_p \right\}$

Cotangent space $T_p^*U = \text{dual of } T_pU$

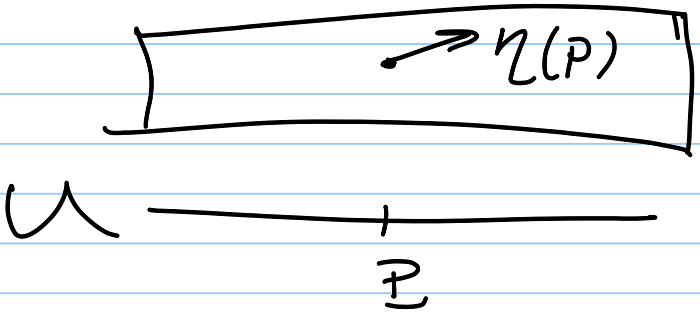
$= \text{span}\{ dx_1|_p, \dots, dx_n|_p \}$ such that

$$dx_i(p)\left(\frac{\partial}{\partial x_j}\Big|_p\right) = \delta_{ij}, \text{ for all } i,j = 1,\dots,n.$$

$$\begin{tikzcd}
T^*U = \bigcup_{p \in U} T^*_p U \arrow[d, "\pi"'] & \ni (p, \eta) \arrow[d, mapsto] \\
U & p
\end{tikzcd}$$

As in the case of tangent bundle the cotangent bundle is a smooth manifold of dimension $2n$ and the projection map π is smooth.

A 1-form on U is a smooth section of the map $\pi : T^*U \to U$: $\eta : U \to T^*U$ such that $\pi \circ \eta : U \to U$ is identity.



$\eta(p) = a_1(p)\, dx_1|_p + \cdots + a_n(p)\, dx_n|_p$

$a_i : U \to \mathbb{R}$ smooth functions

$\underline{\underline{\text{Ex}}}$: $M = \mathbb{R}^3$, $\omega(x,y,z) = x^2y\, \underline{dx} - 3e^xz\, \underline{dy} + dz$ is a smooth one form on $\mathbb{R}^3$.

If $X(x,y,z) = x \underline{\frac{\partial}{\partial x}}|_p - 2 \underline{\frac{\partial}{\partial y}}|_p - y \frac{\partial}{\partial z}|_p$, then the composition $\omega(X)(x,y,z)$ is a function on $\mathbb{R}^3$ given by

$\omega(X)(x,y,z) = x^3 y + 6e^x z - y$

k-form on $U \subseteq \mathbb{R}^n$ $\underline{dx_i} = \underline{dx_i}|_p$

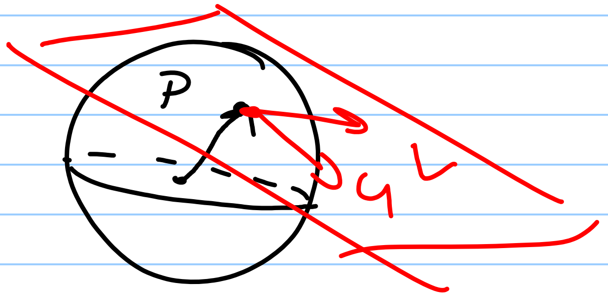
$f(p)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$

In general, a k-form on U has the form

$\omega = \sum_I f_I(p)\, dx_I$, $I = (i_1 < i_2 < \cdots < i_k)$ $1 \leq i_j \leq n$

$dx_I \doteq dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

Örnek: S^2 unit sphere in $\mathbb{R}^3$. Let ω be the 2-form on S^2 given by



$\omega(p)(u,v) \doteq \underline{(u \times v)} \cdot p$

$u, v \in T_p S^2$

$p = (x,y,z)$, $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

$\omega(p)(u,v) = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1) \cdot (x,y,z)$

$\Rightarrow \omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$