Note Title 18.02.2020

$U \subseteq \mathbb{R}^n$ $\omega$ $k$-form on $U$

$\omega = \sum f_I \, dx_I$ $I = (i_1, \dots, i_k)$ $1 \leq i_1 < i_2 < \dots < i_k \leq n$

$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$f_I : U \to \mathbb{R}$ smooth function

<u>Wedge product</u>

$(f_I \, dx_I) \wedge (g_J \, dx_J) = f_I g_J \, dx_I \wedge dx_J$

and for general forms we extend this definition linearly.

If $\omega$ is a $k$-form and $\nu$ is an $l$-form then $\omega \wedge \nu = (-1)^{kl} \nu \wedge \omega$.

$(dx_{i_1} \wedge - \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge - \wedge dx_{j_l})$

$$= \underbrace{(-1)^k (-1)^k \dots (-1)^k}_{l} \, dx_{j_1} \wedge \dots \wedge dx_{j_l} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

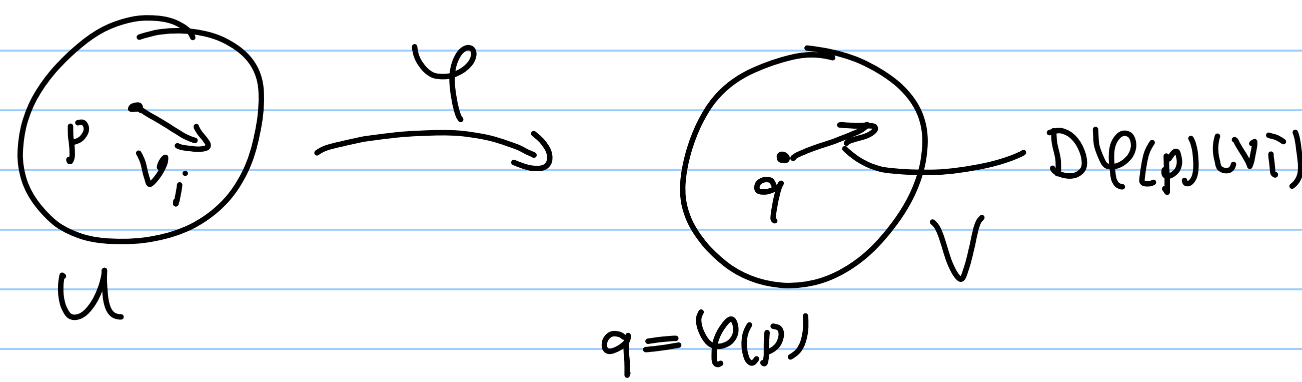
<u>Exterior Derivation</u>

$f_I \, dx_I$ $k$-form, $d(f_I \, dx_I) \doteq df_I \wedge dx_I$

**Fact:** $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\,\omega\wedge d\eta$, where $\omega$ is a $k$-form and $\eta$ is a $l$-form.

**Pull back:** $\varphi : U \to V$, $U\subseteq\mathbb{R}^n$, $V\subseteq\mathbb{R}^m$, $\varphi$ smooth function. Let $\omega$ be a $k$-form on $V$. Then $\varphi^*(\omega)$ is a $k$-form on $U$ defined by

$$\varphi^*(\omega)(v_1,\dots,v_k) \doteq \omega\big(D\varphi(p)(v_1),\dots,D\varphi(p)(v_k)\big)$$




$\varphi^*$ commutes with $d$ and wedge product.

- $\varphi^*(d\omega) = d(\varphi^*(\omega))$
- $\varphi^*(\omega\wedge\eta) = \varphi^*(\omega)\wedge\varphi^*(\eta)$.

**Practically we compute $\varphi^*$ as follows:**

$$\varphi : U \to V \qquad \mathbb{R}^n \quad x_1, \dots, x_n$$
$$U \subseteq \mathbb{R}^n, \quad V \subseteq \mathbb{R}^m \qquad \mathbb{R}^m \quad y_1, \dots, y_m$$

$\omega \in \Omega^k(V)$, $\omega = \sum f_I \, dy_I$

$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m) = (y_1, \dots, y_m)$

$y_i = \varphi_i(x_1, \dots, x_n)$, $dy_i = d\varphi_i$

Hence, $dy_I = d\varphi_I = d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_k} \in \Omega^k(U)$.

**Example:** $\varphi : \underset{u,v}{\mathbb{R}^2} \to \underset{x,y,z}{\mathbb{R}^3}$, $\varphi(u,v) = (\underbrace{u+v}_{x}, \underbrace{u \cdot v}_{y}, \underbrace{v^2+u}_{z})$

$\omega \in \Omega^2(\mathbb{R}^3)$, $\omega = x \, dx \wedge dy - (z+y) \, dy \wedge dz$
$+ 3 \, dz \wedge dx.$

$\varphi^*(\omega)$ ?

$x = x(u,v) = u+v$, $dx = du + dv$
$y = y(u,v) = u \cdot v$, $dy = v \, du + u \, dv$

$z = z(u,v) = v^2 + u$, $dz = du + 2v \, dv$

So, $\varphi^*(\omega) = x \, (du + dv) \wedge (v \, du + u \, dv)$

$- (z+y) (v \, du + u \, dv) \wedge (du + 2v \, dv)$
$+ 3 \, (du + 2v \, dv) \wedge (du + dv)$

$= xu \, du \wedge dv - xv \, du \wedge dv$

$-(z+y) (2v^2 \, du \wedge dv - u \, du \wedge dv)$
$+ 3 \, (du \wedge dv - 2v \, du \wedge dv)$

$= (xu - xv - (z+y)2v^2 + (z+y)u + 3 - 2v)\, du \wedge dv$

$= \left[(u+v)u - (u+v)v - (v^2u+uv)2v^2 \right.$
$\left. + (v^2u+uv)u + 3 - 2v\right] du \wedge dv.$

<u>Example:</u> $S^1 = \{(x,y,0) \mid x^2+y^2=1\} \subseteq \mathbb{R}^3$

$\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\}).$

$\varphi : \mathbb{R}^3 \setminus S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\} \quad (x,y,z) \longmapsto (x^2+y^2-1, z).$

$u = x^2+y^2-1, \; v = z, \; \omega = \dfrac{u\,dv - v\,du}{u^2+v^2}$

$du = 2x\,dx + 2y\,dy, \; dv = dz.$

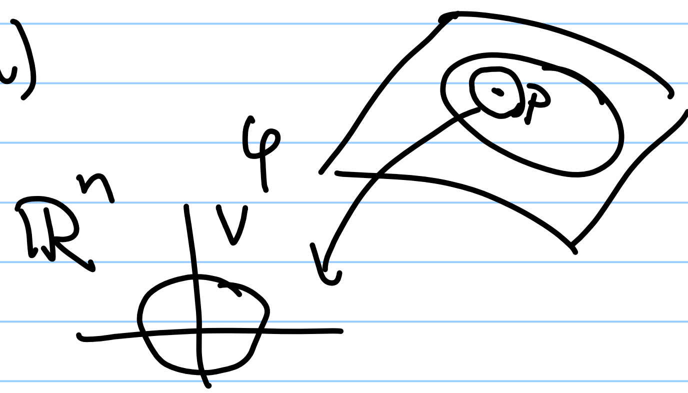
$\varphi^*(\omega) = \dfrac{(x^2+y^2-1)\,dz - z(2x\,dx + 2y\,dy)}{(x^2+y^2-1)^2 + z^2}$

$\varphi^*(\omega) \in \Omega^1(\mathbb{R}^3 \setminus S^1).$

<u>Forms on Manifolds</u>

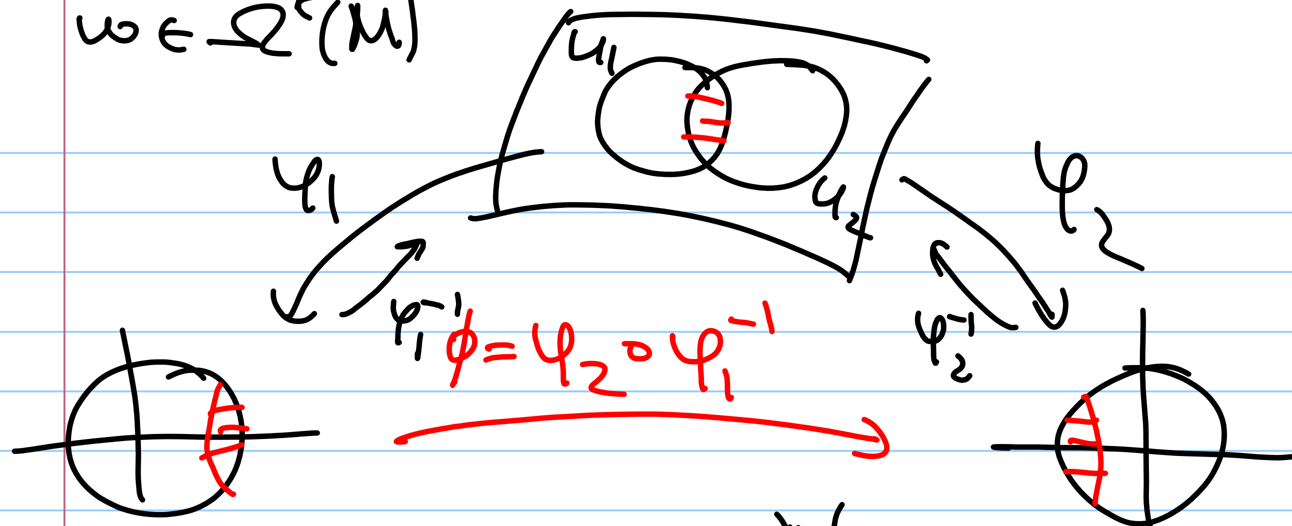
$M^n$ smooth manifold, $U \subseteq M$ open subset

$\omega \in \Omega^k(U)$




$\omega = \varphi^*(\eta)$
$\eta \in \Omega^k(V).$

$\omega \in \Omega^k(M)$




$$(\varphi_1^{-1})^*(\omega) = \phi^*\left((\varphi_2^{-1})^*(\omega)\right)$$

<u>Example:</u> $\phi : \underset{t_1, t_2}{\mathbb{R}^2} \longrightarrow \underset{(x_1, y_1, x_2, y_2)}{\mathbb{R}^4}$

$$\phi(t_1, t_2) = (\underbrace{\cos t_1}_{x_1}, \underbrace{\sin t_1}_{y_1}, \underbrace{\cos t_2}_{x_2}, \underbrace{\sin t_2}_{y_2})$$

$\phi(\mathbb{R}^2) = S^1 \times S^1 \qquad \omega_i \in \Omega^1(\mathbb{R}^4)$

$$\omega_1 = \frac{x_1 dy_1 - y_1 dx_1}{x_1^2 + y_1^2}, \quad \omega_2 = \frac{x_2 dy_2 - y_2 dx_2}{x_2^2 + y_2^2}$$

$$\varphi^* \omega_1 = (\cos t_1)(\cos t_1)\, dt_1 - (\sin t_1)(-\sin t_1)\, dt_1$$
$$= dt_1$$

Similarly, $\phi^* \omega_2 = dt_2$.

So $\varphi^*(\omega_1 \wedge \omega_2) = dt_1 \wedge dt_2$.

<u>Lemma:</u> $d_k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$, $k \geq 0$

Then $d_{k+1} \circ d_k = 0$.

# Orientation of Manifolds

**Orientation of Vector spaces:** Let $\beta$ and $\beta'$ be two ordered bases for a vector space $V$ (finite dim'l.)

$\beta = (v_1, \ldots, v_n)$, $\beta' = (u_1, \ldots, u_n)$.

We say that $\beta$ and $\beta'$ are equivalent if the base change matrix $[I]_{\beta}^{\beta'}$ has positive determinant.

$$u_1 = a_{11} v_1 + a_{12} v_2 + \cdots + a_{1n} v_n$$
$$\vdots$$
$$u_i = a_{i1} v_1 + a_{i2} v_2 + \cdots + a_{in} v_n$$
$$\vdots$$
$$u_n = a_{n1} v_1 + a_{n2} v_2 + \cdots + a_{nn} v_n$$

$$[I]_{\beta}^{\beta'} = [a_{ij}]$$

$$[I]_{\beta'}^{\beta} = \left([I]_{\beta}^{\beta'}\right)^{-1}$$

This relation on the set of all ordered bases of $V$ is an equivalence relation:

i) $\beta \sim \beta' \Rightarrow \beta' \sim \beta$ (Symmetric.)

ii) $\beta \sim \beta$, $[I]_\beta^\beta = Id$, $\det([I]_\beta^\beta) = 1$

(~ is reflexive)

iii) $\beta \sim \beta'$ and $\beta' \sim \beta''$, then

$$[I]_\beta^{\beta''} = [I]_{\beta'}^{\beta''}[I]_\beta^{\beta'} \Rightarrow \det([I]_\beta^{\beta''}) > 0.$$

(~ is transitive)

<u>Remark:</u> Orientation on a 0-dimensional vector space is just an assignment of $\pm$ sign.

An orientation on a vector space $V$ is a choice of an equivalence class of the equivalence relation defined above.

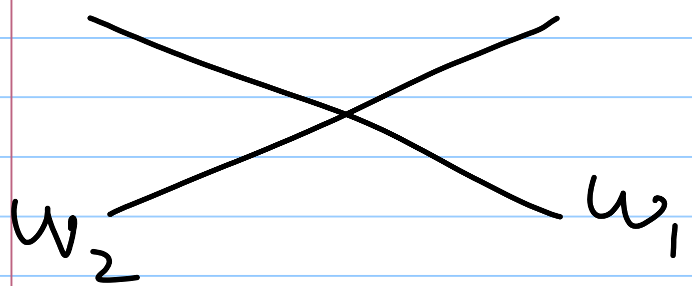
On a vector space there are two orientations.

An orientation of a subspace $W$ of $V$ is just an orientation of $W$ as a vector space.

Let $W_1$ and $W_2$ be two <u>oriented</u> subspaces of an oriented space $V$ so that

$W_1 + W_2 = V$ and $\dim V = \dim W_1 + \dim W_2$.

In particular, $W_1 \cap W_2 = \{0\}$.




The orientation of the intersection $\{0\} = W_1 \cap W_2$ is defined as follows:

let $\beta_1 = (u_1, \dots, u_k)$ and $\beta_2 = (u_{k+1}, \dots, u_n)$ be oriented bases for $W_1$ and $W_2$.

$W_1 + W_2 = V \Rightarrow \beta = (u_1, \dots, u_k, u_{k+1}, \dots, u_n)$ is a basis for $V$. If the orientation on $V$ given by $\beta$ is the same as the orientation of $V$ then the sign of the intersection $\{0\}$ is defined to be "+".

Otherwise, it is defined to be "−".

<u>Ex</u> $V = \mathbb{R}^4 = (e_1, e_2, e_3, e_4)$

$W_1 = \text{span}(e_1, -e_2)$, $W_2 = (e_3, e_4)$.

$W_1 \cap W_2 = \{0\}$ $(e_1, -e_2, e_3, e_4) \overset{?}{\sim} (e_1, e_2, e_3, e_4)$

No!

Hence, the orientation on $W_1 \cap W_2$ is "$-$".

<u>Remark</u>: Note that the orientation on $W_1 \cap W_2$ may be different than the orientation on $W_2 \cap W_1$.

<u>Example</u>: $V = \mathbb{R}^2 = (e_1, e_2)$

$W_1 = (-e_1)$, $W_2 = (e_2)$

$W_1 \cap W_2 \to$ "$+$", but $W_2 \cap W_1 \to$ "$-$"

Now consider the case, where $\dim W_1 + \dim W_2 > \dim V$, and $W_1 + W_2 = V$.

Let $(v_1, v_2, \ldots, v_k)$ be an ordered bases for $W_1 \cap W_2$.

$W_1 = (u_1, \ldots, u_\ell, v_1, \ldots, v_k)$ , $\dim W_1 = k + \ell$

$W_2 = (v_1, \ldots, v_k, u_{\ell+1}, \ldots, u_{n-k})$, $\dim W_2 = n - \ell$

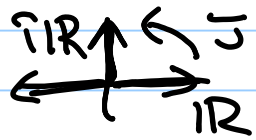
Then $(u_1, \ldots, u_\ell, v_1, \ldots, v_k, u_{\ell+1}, \ldots, u_{n-k})$ is an oriented basis for $V$. If this basis gives the right orientation on $V$ then the orientation of $W_1 \cap W_2$ is the one given by $(v_1, \ldots, v_k)$. Otherwise, the orientation on $W_1 \cap W_2$ is given by $(-v_1, v_2, \ldots, v_k)$.

__Orientations on Complex Vector spaces__

$V$ is an $n$-dim'l complex vector space. Then $V$ is a $2n$-dimensional real vector space together with an complex structure $J: V \longrightarrow V$ such that $J^2 = -I$.

__Ex__ $V = \mathbb{C}^n = \mathbb{R}^{2n}$ $\quad J: \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$

$n=1$, $\mathbb{C} = \mathbb{R}^2 = \{(1,0), (0,1)\}$




$$J(1,0) = (0,1) = i\,(1,0).$$
$$J^2(1,0) = i^2(1,0) = (-1,0)$$

__Fact:__ A complex vector space, regarded as a real vector space, has a canonical orientation.

__Proof:__ $\dim_{\mathbb{R}} V = 2n$, $J: V \longrightarrow V$, $J^2 = -Id$.

Let $v_1 \in V$, $v_1 \neq 0$. Then let $v_2 = J v_1$.

If $\dim_{\mathbb{R}} V > 2$ then choose $v_3 \in V \setminus \mathrm{span}\{v_1, v_2\}$

Then let $v_4 = \bar{u} v_3$.

<u>Claim:</u> $\{v_1, v_2, v_3, v_4\}$ is $\mathbb{R}$-linearly independent.

In general if $\dim V = n$, then we can find a basis ($\mathbb{R}$ basis) for $V$ of the form $(v_1, \bar{u}v_1, v_2, \bar{u}v_2, \dots, v_n, \bar{u}v_n)$.

<u>Claim:</u> If $(u_1, \bar{u}u_1, u_2, \bar{u}u_2, \dots, u_n, \bar{u}u_n)$ is another such basis then these two bases define the same orientation.