Last time: V complex vector space of dim n.

$\dim_{\mathbb{R}} V = 2n$, $J : V \to V$, $J^2 = -I$.

Claim: The real vector space V with the complex structure has a canonical orientation.

Proof: Let $v_1 \in V$, $v_1 \neq 0$. Then let $v_2 = Jv_1$.

Claim: v_1 and v_2 are linearly independent.

Proof: Let $\boxed{a_1 v_1 + a_2 v_2 = 0}$, for some $a_1, a_2 \in \mathbb{R}$.

Then $J(a_1 v_1 + a_2 v_2) = J(0) = 0$

$$a_1 J v_1 + a_2 J v_2 = 0 \Rightarrow a_1 v_2 + a_2 J^2 v_1 = 0$$

$$\Rightarrow \boxed{-a_2 v_1 + a_1 v_2 = 0.}$$

$$\begin{array}{rl} \cancel{a_1 v_1} + a_2 v_2 = 0 & / \, a_2 \\ -a_2 \cancel{v_1} + a_1 v_2 = 0 & / \, a_1 \\ \hline + & \end{array}$$

$(a_1^2 + a_2^2) v_2 = 0$ since $v_1 \neq 0$, $v_2 \neq 0$

$\Rightarrow a_1^2 + a_2^2 = 0 \Rightarrow a_1 = a_2 = 0$.

If $\dim_{\mathbb{R}} V = 2n > 2$ then choose $v_3 \in V$, which is not in the linear span of v_1 an

v_2. Let $v_4 = i v_3$.

Claim $\{v_1, v_2, v_3, v_4\}$ is $\mathbb{R}$-linearly independent

Proof: Suppose that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$ for some $a_i \in \mathbb{R}$, $i = 1, \ldots, 4$. Take i of this expression to get the system:

$$
\begin{array}{l|l}
a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0 & a_3 \\
+\ a_1v_2 - a_2v_1 + a_3v_4 - a_4v_3 = 0 & -a_4 \\
\hline
\end{array}
$$

$$(a_1a_3 + a_2a_4)v_1 + (-a_1a_4 + a_2a_3)v_2 + (a_3^2 + a_4^2)v_3 = 0$$

$\Rightarrow a_3^2 + a_4^2 = 0 \Rightarrow a_3 = a_4 = 0$

$\Rightarrow a_1v_1 + a_2v_2 = 0 \Rightarrow a_1 = a_2 = 0$

This way we see that the set

$\{v_1, \underbrace{iv_1}_{v_2}, v_3, \underbrace{iv_3}_{v_4}, \ldots, v_{2n-1}, \underbrace{iv_{2n-1}}_{v_{2n}}\}$ is linear $\mathbb{R}$-independent.

Claim: $\{v_1, v_3, \ldots, v_{2n-1}\}$ is a $\mathbb{C}$-basis for V.

Proof: If $v \in V$ then $v = \sum_{i=1}^{2n} a_i v_i$, $a_i \in \mathbb{R}$.

$v = a_1v_1 + a_2v_2 + \cdots + a_{2n-1}v_{2n-1} + a_{2n}v_{2n}$

$v_2 = Jv_1 = iv_1$

$\Rightarrow v = a_1 v_1 + i a_2 v_1 + \dots + a_{2n-1} v_{2n-1} + i a_{2n} v_{2n-1}$

$= (a_1 + i a_2) \underline{\underline{v_1}} + \dots + (a_{2n-1} + i a_{2n}) \underline{\underline{v_{2n-1}}}$

$\Rightarrow \{v_1, v_3, \dots, v_{2n-1}\}$ spans V as a cx. vector space.

Let $\{u_1, u_2 = Ju_1, \dots, u_{2n-1}, u_{2n} = Ju_{2n-1}\}$ be another such basis for V.

<u>Claim</u>: $\{v_i\}$ and $\{u_i\}$ induce the same orientation.

<u>Proof</u>: $\{v_1, v_3, \dots, v_{2n-1}\} = \beta$ and $\{u_1, u_2, \dots, u_{2n-1}\} = \beta'$ are two complex basis for the complex vector space V. Let $A = [I]_{\beta}^{\beta'} \in \mathbb{C}^{n \times n}$

Let $A = (a_{\tilde{i}\tilde{j}})$ and set $\alpha_{\tilde{i}\tilde{j}} = \begin{pmatrix} Re(a_{\tilde{i}\tilde{j}}) & -Im(a_{\tilde{i}\tilde{j}}) \\ Im(a_{\tilde{i}\tilde{j}}) & Re(a_{\tilde{i}\tilde{j}}) \end{pmatrix}$

The base change matrix from $\{v_1, \dots, v_{2n}\}$ to $\{u_1, \dots, u_{2n}\}$ is the $2n \times 2n$ real matrix obtained from A by replacing each $a_{\tilde{i}\tilde{j}}$ with 2×2-real matrix $\alpha_{\tilde{i}\tilde{j}}$.

Ex:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "a"] \arrow[d, phantom, "\shortparallel"] & \mathbb{C} \arrow[d, phantom, "\shortparallel"] \\ \mathbb{R}^2 \arrow[r, "A"'] & \mathbb{R}^2 \end{tikzcd}, \quad a = b + ic$$

$$\operatorname{Re} a = b, \ \operatorname{Im} a = c.$$

$$a \cdot z = (b+ic)(x+iy) = (bx - cy) + i(cx + by)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \overset{}{\underset{A}{\longmapsto}} \begin{bmatrix} bx - cy \\ cx + by \end{bmatrix} \qquad A = \begin{bmatrix} b & -c \\ c & b \end{bmatrix}$$

<u>must show:</u> $\det A > 0$.

In $n=1$ case, $\det A = b^2 + c^2 > 0$.

$$\underline{A_{\mathbb{C}}} = (a_{ij}) \longleftrightarrow A = (\alpha_{ij})$$

$$\lambda = \mu + i\epsilon \longleftrightarrow \underline{\begin{pmatrix} \mu & -\epsilon \\ \epsilon & \mu \end{pmatrix}}$$

$$\begin{tikzcd} A_{\mathbb{C}} \arrow[d, squiggly] & & A \arrow[ll] \arrow[d, squiggly] \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \underbrace{\det A_{\mathbb{C}}}_{\theta} \end{pmatrix} \arrow[rr, leftrightarrow, "n\times n"'] & & \begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & \operatorname{Re}\theta & -\operatorname{Im}\theta \\ 0 & & & \operatorname{Im}\theta & \operatorname{Re}\theta \end{pmatrix} \end{tikzcd}$$

So, $\det A = |\det A_{\mathbb{C}}|^2 > 0$.

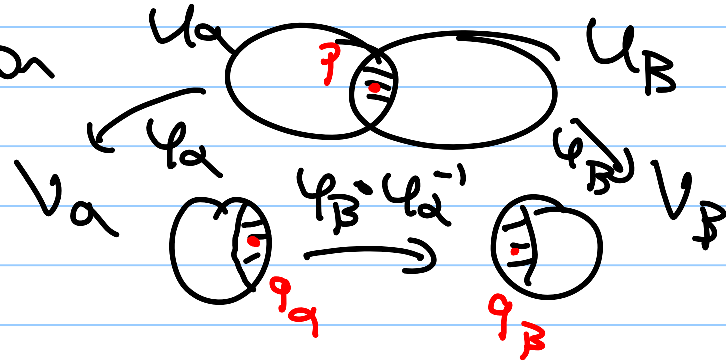
Orientations on Manifolds

$U \subseteq \mathbb{R}^n$ open subset, $T_*U = U \times \mathbb{R}^n$

An orientation on U is a choice of an orientation on the $\mathbb{R}^n$ part of T_*U.

Let $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$ be an atlas for a smooth manifold M. Then $T_*U_\alpha \simeq T_*V_\alpha = V_\alpha \times \mathbb{R}^n$

Put an orientation on each V_α. If each transition function



If the linear map

$$D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(q_\alpha)} : T_{q_\alpha}V_\alpha \longrightarrow T_{q_\beta}V_\beta$$

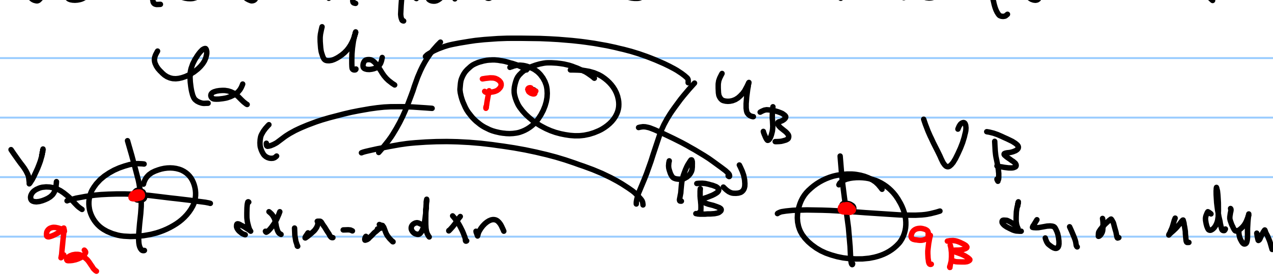
preserves the orientation for all α, β and $q_\alpha = \varphi_\alpha(p)$, $p \in U_\alpha \cap U_\beta$, then we say that the orientations on V_α's put an orientation on M. In this case, we say that M is orientable. Moreover, each choice of orientation makes M an oriented manifold.

If M does not admit any orientation then we say that M is not orientable.

Proposition: A smooth manifold is orientable if and only if there is an n-form ω on M ($\dim M = n$) so that $\omega(p) = f\, dx_1 \wedge \cdots \wedge dx_n$ is not zero at any $p \in M$.

Proof: Suppose such ω exists. For any point $p \in M$ and any basis $\{v_1, \ldots, v_n\}$ for T_pM we check if $\omega(p)(v_1, \ldots, v_n) > 0$. If it is $+$ choose this basis as the orientation at p. This orients the smooth manifold.

If M is oriented then we can choose nowhere zero n-form ω on M as follows:



$$L = D(\varphi_\beta \circ \varphi_\alpha^{-1})(q_\alpha)$$

Choose x_i's so that $(dx_1 \wedge \cdots \wedge dx_n)(v_1, \ldots, v_n) > 0$.
Since transition functions preserves local orientation we see that

$$L^*(dy_1 \wedge \cdots \wedge dy_n) = \lambda \, dx_1 \wedge \cdots \wedge dx_n, \quad \lambda > 0$$

Choose a partition of unity $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}$ $\rho_\alpha \geq 0$, $\sum_\alpha \rho_\alpha(p) = 1$, which is a finite sum, and $\mathrm{supp}(\rho_\alpha) \subseteq U_\alpha$.

$$\omega = \sum_\alpha \rho_\alpha \varphi_\alpha^*(dx_1 \wedge \cdots \wedge dx_n)$$

$$\omega(v_1, \ldots, v_n) > 0.$$

$T_pM = \mathrm{span}\langle v_1, \ldots, v_n \rangle$ oriented basis.

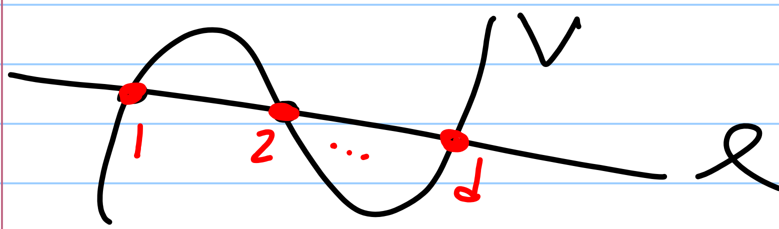
Ex: $\mathbb{C}^n$, $V \subseteq \mathbb{C}^n$; $V = (f = 0)$, $f : \mathbb{C}^n \to \mathbb{C}$ poly. map

Assume $0 \in \mathbb{C}$ is a regular value for f, then V is a $n-1$-dim'l complex submanifold.
Let ℓ be a complex line in $\mathbb{C}^n$. Then $V \cap \ell$ has at most d-points, where $d = \deg f$.

$\ell = \mathbb{C}$, $V: f(z_1, \dots, z_n) = 0$

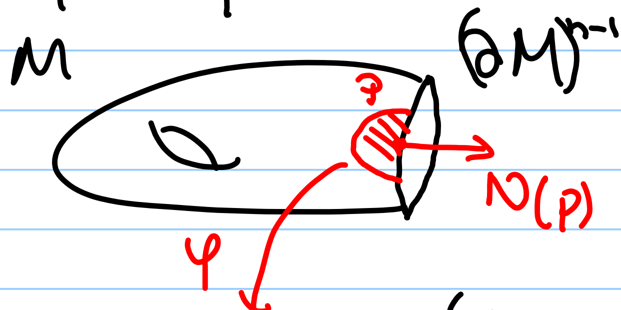
$\ell : z_2 = z_3 = \dots = z_n = 0$

$V \cap \ell = ?$ $\quad f(z_1, 0, \dots, 0) = 0 \Rightarrow$ This equation has d solutions (counted with multiplicity)

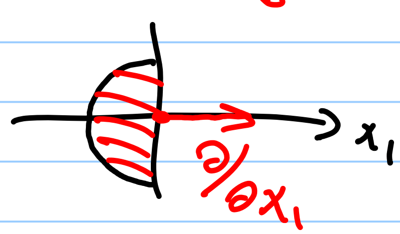


<u>Theorem</u> (Stokes' Theorem)

Let M be a compact smooth oriented manifold. Then ∂M is a smooth orientable manifold of dimension $n-1$.



$T_p \partial M = \langle v_1, \dots, v_{n-1} \rangle$ basis

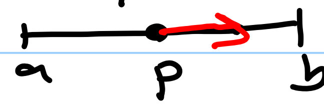


$\{v_1, \dots, v_{n-1}\}$ gives the right orientation to ∂M at p if $\{\vec{N}(p), v_1, \dots, v_{n-1}\}$ gives the right orientation to M at p.

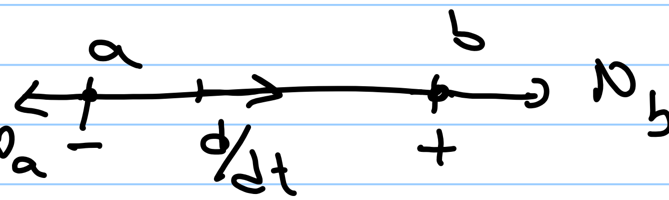
In this case, for any $n-1$-form ω on M we have $\int_{M^n} d\omega = \int_{\partial M} \omega$.

$T_p M = \mathbb{R} = \left\langle \frac{d}{dt} \right\rangle$

<u>Example:</u> $M = [a,b] \subseteq \mathbb{R}$



$\partial M = \{\bar{a}, \overset{+}{b}\}$



$\omega = f$ 0-form on $M = [a,b]$

$d\omega = f'(t)\,dt$ 1-form

$$\int_M d\omega = \int_{[a,b]} f'(t)\,dt = f(b) - f(a) = \int_{\partial M = \{a^-, b^+\}} f$$