Disk ve Kürenin Hacimleri

$$Vol(D^n(r)) = \int_{D^n(r)} dx_1 \wedge \cdots \wedge dx_n = \int_{D^n(r)} dx_1 \cdots dx_n$$

$$D^n(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$$

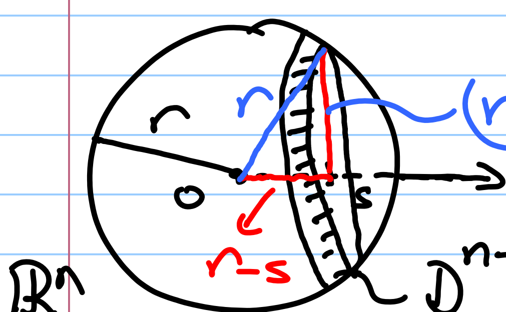
For any integer $n > 0$ let $Vol(D^n(r)) = r^n \cdot A_n$.

So $A_n = \dfrac{Vol(D^n(r))}{r^n}$.

$$Vol(D^1(r)) = \text{Length}([-1,1]) = 2 \Rightarrow A_1 = \frac{2}{1^1} = 2.$$

$$Vol(D^2(r)) = \pi r^2 \Rightarrow A_2 = \frac{\pi r^2}{r^2} = \pi.$$

$$Vol(D^n(r)) = 2\int_0^r \underbrace{Vol(D^{n-1}(\sqrt{r^2-s^2}))}_{= (r^2-s^2)^{\frac{n-1}{2}} A_{n-1}} ds$$

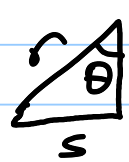


$(r^2-s^2)^{1/2}$

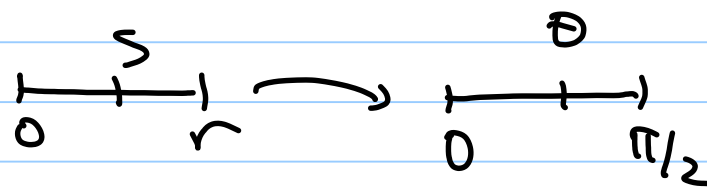
$D^{n-1}(r^2-s^2)^{1/2}$

$$Vol(D^n(r)) = 2A_{n-1}\int_0^r \left(\sqrt{r^2-s^2}\right)^{n-1} ds$$

Let $s = r\sin\theta$



$\sin\theta = \dfrac{s}{r}$



$r^2 - s^2 = (\cos^2\theta) r^2$

$ds = r\cos\theta \, d\theta$

Hence, we get

$$\mathrm{Vol}(D^n(r)) = 2A_{n-1}\int_0^{\pi/2} (\cos\theta)^n r^n \, d\theta$$

$$= 2r^n A_{n-1} \int_0^{\pi/2} \cos^n\theta \, d\theta$$

Let $B_n \doteq \int_0^{\pi/2} \cos^n\theta \, d\theta$.

$$B_1 = \int_0^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi/2} = 1.$$

$$B_2 = \int_0^{\pi/2} \cos^2\theta \, d\theta = \pi/4.$$

For $n \geq 2$, let $v = (\cos\theta)^{n-1}$, $du = \cos\theta \, d\theta$

$u = \sin\theta$

$dv = (n-1)\cos^{n-2}\theta \, (-\sin\theta)$

$$B_n = \int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} v \, du$$

$$= uv\Big|_0^{\pi/2} - \int_0^{\pi/2} u \, dv$$

$$= (\cos\theta)^{n-1}\sin\theta\Big|_0^{\pi/2} + \int_0^{\pi/2} \sin^2\theta \cos^{n-2}\theta \, (n-1) \, d\theta$$

$$= 0 + \int_0^{\pi/2} (1-\cos^2\theta)\cos^{n-2}\theta \, (n-1) \, d\theta$$

$$= \left[\int_0^{\pi/2} \cos^{n-2}\theta \, d\theta - \int_0^{\pi/2} \cos^n\theta \, d\theta\right](n-1)$$

$$B_n = (n-1) B_{n-2} - (n-1) B_n$$

$$n B_n = (n-1) B_{n-2} \implies B_n = \frac{n-1}{n} B_{n-2}.$$

$$B_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!} \quad \text{and} \quad B_{2n} = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}$$

We also have $A_n / A_{n-1} = 2B_n$.

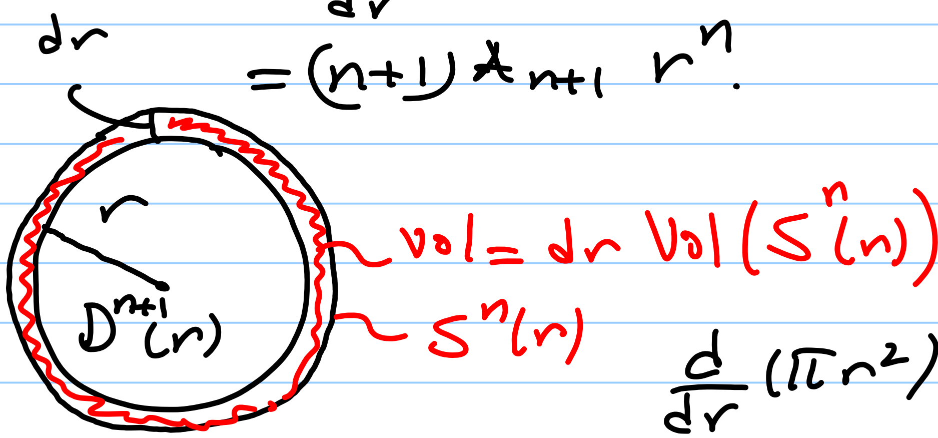
Using this we get

$$A_{2n+1} = \frac{2^{n+1}\pi^n}{1\cdot 3 \cdot \cdots \cdot (2n+1)} \quad \text{and} \quad A_{2n} = \frac{\pi^n}{n!}$$

So, $\mathrm{Vol}(D^n(r)) = A_n r^n$ is computed.

On the other hand, we have

$$\begin{aligned} \mathrm{Vol}(S^n(r)) &= \frac{d}{dr}\mathrm{Vol}(D^{n+1}(r)) \\ &= \frac{d}{dr}(A_{n+1} r^{n+1}) \\ &= (n+1) A_{n+1} r^n. \end{aligned}$$



$$\frac{d}{dr}(\pi r^2) = 2\pi r$$

<u>Example:</u> $\mathrm{Vol}(D^4(r)) = \frac{\pi^2 r^4}{2}$, $\mathrm{Vol}(S^3(r)) = 2\pi^2 r^3$.

Example: $\mathbb{R}^n \setminus \{0\}$, $\omega_{S^{n-1}} \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$

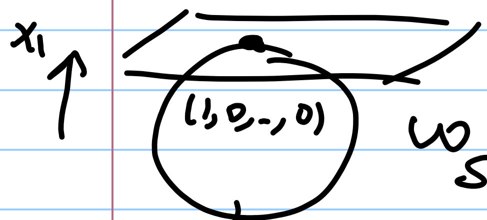
$$\omega_{S^{n-1}} \doteq \sum_{i=1}^{n} (-1)^{i-1} x_i \frac{dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n}{(x_1^2 + \cdots + x_n^2)^{n/2}}$$

$d\omega_{S^{n-1}} = 0$ (Exercise)

$A \in SO(n)$, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Exercise: $A^* \omega_{S^{n-1}} = \omega_{S^{n-1}}$, so that $\omega_{S^{n-1}}$ is $SO(n)$ invariant.

Let's compute this for on the basis vectors $\{\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$ of $T_{(1,0,\dots,0)} S^{n-1}$.



$$\omega_{S^{n-1}}\left(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right) = 1.$$

$$\int_{S^{n-1}} \omega_{S^{n-1}} = \mathrm{Vol}(S^{n-1}) = n A_n$$

Now define $\omega_{0,\mathbb{R}^n} = \dfrac{\omega_{S^{n-1}}}{n A_n}$.

Then $\displaystyle\int_{S^{n-1}} \omega_{0,\mathbb{R}^n} = 1.$

Soon, we'll see that $\omega_{0,\mathbb{R}^n}$ is the only nontrivial $n-1$-form on $\mathbb{R}^n \setminus \{0\}$.

<u>Remark:</u> Let $M \subseteq \mathbb{R}^n$ be an $n-1$ dimensional smooth closed oriented manifold, so that $0 \notin M$.

$M^{n-1} = \partial V^n$ for some smooth compact manifold V^n.

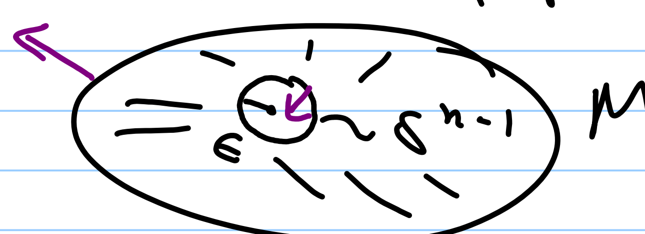
$$\int_M \omega_{0,\mathbb{R}^n} = \frac{1}{nA_n} \int_M \omega_{S^{n-1}}$$

If $0 \notin V^n$, then from Stokes' Theorem

$$\int_M \omega_{S^{n-1}} = \int_{\partial V^n} \omega_{S^{n-1}} = \int_{V^n} \underbrace{d\omega_{S^{n-1}}}_{=0 \text{ since } \omega_{S^{n-1}} \text{ is closed}} = 0.$$

So $\int_M \omega_{0,\mathbb{R}^n} = 0$.

If $0 \in V^n$, then choose a small sphere S_ϵ^{n-1} around 0.



$V^n \setminus \mathring{S}_\epsilon^{n-1}$ = compact smooth manifold with boundary

$\partial(V^n \setminus S_\epsilon^{n-1}) = M \,\dot\cup\, (-S_\epsilon^{n-1})$. In particular, $0 \notin V^n \setminus S_\epsilon^{n-1}$, and thus

$$0 = \int_{V^n \setminus S_\epsilon^{n-1}} \overbrace{d\omega_{0,\mathbb{R}^n}}^{=0} = \int_{\partial(V^n \setminus S_\epsilon^{n-1}) = M \setminus (-S_\epsilon^{n-1})} \omega_{0,\mathbb{R}^n}$$

$$= \int_M \omega_{0,\mathbb{R}^n} - \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

$$\Rightarrow \int_M \omega_{0,\mathbb{R}^n} = \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

In particular, if we take $M = S_1^{n-1}$, then

$$1 = \int_{S_1^{n-1}} \omega_{0,\mathbb{R}^n} = \int_{S_\epsilon^{n-1}} \omega_{0,\mathbb{R}^n}$$

Hence, $\int_M \omega_{0,\mathbb{R}^n} = 1$.

Summary: $M^{n-1} = \partial V^n$, $V^n \subseteq \mathbb{R}^n$ smooth compact manifold.

Then

$$\int_M \omega_{0,\mathbb{R}^n} = \begin{cases} 0 & \text{if } 0 \notin V^n \\ 1 & \text{if } 0 \in V^n. \end{cases}$$

$\omega_{0,\mathbb{R}^n}$ is called the "linking form" of the origin in $\mathbb{R}^n$.

Special Forms on Complex Manifolds:

$\mathbb{C}^n = \mathbb{R}^{2n}$ $\quad z_1, \dots, z_n$, $z_k = x_k + i y_k$

$z_k \quad x_k, y_k$

$dz_k = dx_k + i\, dy_k$. Consider the 2-form on $\mathbb{C}^n = \mathbb{R}^{2n}$, given by

$$\omega = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k = \sum_{k=1}^{n} dx_k \wedge dy_k,$$

where $\bar{z}_k = x_k - i y_k$ and $d\bar{z}_k = dx_k - i\, dy_k$.

Exercise: $\omega \in \Omega^2(\mathbb{C}^n) = \Omega^2(\mathbb{R}^{2n})$.

For any $0 \leq l \leq n$ (integer) we have

$$\omega^l = \left(\sum_{k=1}^{n} dx_k \wedge dy_k\right)^l = \left(\frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k\right)^l$$

$$= \left(\frac{i}{2}\right)^l l! \sum_{1 \leq k_1 < k_2 < \dots < k_l \leq n} dz_{k_1} \wedge d\bar{z}_{k_1} \wedge \dots \wedge dz_{k_l} \wedge d\bar{z}_{k_l}$$

Let $V \subseteq \mathbb{C}^n$ be an l-complex dimensional subspace of $\mathbb{C}^n$. Let $(w_1, \dots, w_l)$ be a linear coordinate system on V: $w_i : V \longrightarrow \mathbb{C}$ linear map.

Let $L : \overset{w_j}{V} \longrightarrow \overset{z_j}{\mathbb{C}^n}$ be the inclusion map given by the expression

$$(z_1, \dots, z_n) = L(w_1, \dots, w_\ell)$$
$$= (a_{11} w_1 + \dots + a_{\ell 1} w_\ell, \; a_{12} w_1 + \dots + a_{\ell 2} w_\ell, \; \dots, \; a_{1n} w_1 + \dots + a_{\ell n} w_\ell)$$

$$\bar{z}_k = \bar{a}_{1k} \bar{w}_1 + \bar{a}_{2k} \bar{w}_2 + \dots + \bar{a}_{\ell k} \bar{w}_\ell, \quad k = 1, \dots, n.$$

$A = (a_{ij})$ a complex $\ell \times n$-matrix.

Then $L^*(dz_j) = a_{1j} dw_1 + \dots + a_{\ell j} dw_\ell$, and

$L^*(d\bar{z}_j) = \bar{a}_{1j} d\bar{w}_1 + \dots + \bar{a}_{\ell j} d\bar{w}_\ell$.

Then we can compute

$$L^*(dz_{k_1} \wedge d\bar{z}_{k_1} \wedge \dots \wedge dz_{k_\ell} \wedge d\bar{z}_{k_\ell})$$
$$= \overline{\det(A_{k_1 \dots k_\ell})} \det(A_{k_1 \dots k_\ell}) \, dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_\ell \wedge d\bar{w}_\ell$$

$0 \leq$

$$= \|\det(A_{k_1 \dots k_\ell})\|^2 dw_1 \wedge d\bar{w}_1 \wedge \dots \wedge dw_\ell \wedge d\bar{w}_\ell,$$

where $A_{k_1 \dots k_\ell}$ is the submatrix of $A = (a_{ij})$ consisting of the rows $k_1, \dots, k_\ell$.

$\iota^*(\omega^\ell) = C_V \left(\frac{i}{2}\right)^\ell \ell!\, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_\ell \wedge d\bar{w}_\ell$,

when $C_V > 0$ is a constant.

sum of the positive determinants

$w_1 = u_1 + i v_1, \ldots, w_\ell = u_\ell + i v_\ell$

V has a canonical orientation given by

$$\left\{ \frac{\partial}{\partial u_1}, i\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_\ell}, i\frac{\partial}{\partial u_\ell} \right\} = \left\{ \frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n} \right\}$$

$$\iota^*(\omega^\ell)\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial v_\ell}\right)$$

$$= C_V \left(\frac{i}{2}\right)^\ell \ell!\, dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_\ell \wedge d\bar{w}_\ell \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right)$$

$$= C_V \left(\frac{i}{2}\right)^\ell \ell! \left(\frac{2}{i}\right)^\ell du_1 \wedge dv_1 \wedge \cdots \wedge du_\ell \wedge dv_\ell \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \ldots, \frac{\partial}{\partial u_n}, \frac{\partial}{\partial v_n}\right)$$

$$= C_V\, \ell! \cdot 1$$

$$= C_V\, \ell! > 0.$$

<u>Conclusion:</u> If M is an complex ℓ-dim'l smooth manifold of $\mathbb{C}^n$ then the restriction of ω^ℓ to M evaluates positively at any

complex oriented tangent space T_pM of M.
In particular, if $U \subseteq M$ is an open subset with compact closure then
$$\int_U \omega^\ell > 0.$$