Note Title 2020-02-26

$\mathbb{C}^n$; $\omega = \frac{i}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i$

$= \sum_{i=1}^{n} dx_i \wedge dy_i$, $z_j = x_j + i y_j$

$V \subseteq \mathbb{C}^n$ ℓ-dim'l cx. subspace

$w_1, \dots, w_\ell$, $w_j = u_j + i v_j$

$\iota : V \hookrightarrow \mathbb{C}^n$ inclusion map

$\iota^* \omega \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial u_\ell}, \frac{\partial}{\partial v_\ell}\right) > 0.$

Corollary: $\mathbb{C}^n$ has no closed smooth complex submanifold of positive dimension.

Proof: Let $M^\ell \subseteq \mathbb{C}^n$ closed complex submanifold of dimension $\ell > 0$. Then, by above

$\int_{M^\ell} \omega^\ell > 0$. On the other hand, the form ω can be written as

$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i = d\left(\sum_{i=1}^{n} x_i dy_i\right) = d\eta$, where

$\eta = \sum_{i=1}^{n} x_i dy_i$. So, ω is exact.

Hence, $\omega^\ell = \omega \wedge \omega^{\ell-1} = d\eta \wedge \omega^{\ell-1} \overset{\overset{0}{\shortparallel}}{=} d(\eta \wedge \omega^{\ell-1})$, provided that $\ell > 0$. However, in this case

$$0 < \int_M \omega^\ell = \int_M d(\eta \wedge \omega^{\ell-1}) \underset{\text{Stokes'}}{=} \int \eta \wedge \omega^{\ell-1} = 0, \text{ a}$$

$\partial M = \emptyset$

contradiction.

Hence, $\ell = \dim M$ must be zero. ∎

Example: Hence, $\mathbb{CP}^n$, $n \geq 1$ and $S^1 \times S^3$ (with its complex structure) do not admit any embedding into some $\mathbb{C}^N$.

Forms on Complex Projective Spaces

$S^2 = \mathbb{CP}^1 \quad x, y, z$

$\cap$

$\mathbb{R}^3 \qquad \omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

$\omega \in \Omega^2(\mathbb{R}^3)$, $\quad \displaystyle\int_{S^2} \omega = 4\pi.$

$\varphi^{-1} : \mathbb{R}^2 \to S^2 \setminus \{(0,0,-1)\} \subseteq \mathbb{R}^3$

$$\varphi^{-1}(r,s) = \Big(\underbrace{\frac{2r}{1+r^2+s^2}}_{x}, \underbrace{\frac{2s}{1+r^2+s^2}}_{y}, \underbrace{\frac{1-r^2-s^2}{1+r^2+s^2}}_{z} \Big),$$

$$x = \frac{2r}{1+r^2+s^2}, \quad y = \frac{2s}{1+r^2+s^2}, \quad z = \frac{1-r^2-s^2}{1+r^2+s^2}$$

$$dx = \frac{2(1-r^2+s^2)dr - 4rs\,ds}{(1+r^2+s^2)^2}$$

$$dy = \frac{2(1-s^2+r^2)ds - 4rs\,dr}{(1+r^2+s^2)^2}, \quad d\rho = -\frac{4r\,dr + 4s\,ds}{(1+r^2+s^2)^2}$$

$$\Rightarrow (\varphi^{-1})^* \omega = 4\frac{dr \wedge ds}{(1+r^2+s^2)^2} = 2i\frac{dz \wedge d\bar{z}}{(1+\|z\|^2)^2},$$

where $z = r + is$.

$w = \frac{1}{z}$ $\quad \mathbb{CP}^1 = \{[z_0 : z_1]\}$ $\quad \frac{z_1}{z_0} = z, \ \frac{z_0}{z_1} = w$

$$(\varphi^{-1})^* \omega = 2i\frac{dw \wedge d\bar{w}}{(1+\|w\|^2)^2}.$$

$\frac{1}{4}\omega$ is called the Fubini–Study 2-form on $\mathbb{CP}^1$ and denoted as ω_{FS}.

$$\int_{\mathbb{CP}} \omega_{FS} = \pi.$$

~ o ~

$f : \mathbb{C}^n \to \mathbb{C}$ smooth function.

$$\partial f \doteq \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i, \quad \bar{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

<u>Claim:</u> $\omega_{FS} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+\|z\|^2)^2} = \frac{i}{2}\partial\bar{\partial}\log(1+\|z\|^2)$.

<u>Proof</u>: $\bar{\partial}\log(1+\|z\|^2) = \frac{\partial}{\partial \bar{z}}\log(1+z\bar{z})\,d\bar{z}$

So, $\bar\partial \log(1+\|z\|^2) = \frac{z}{1+z\bar z}\, d\bar z$, hence

$$\partial\bar\partial \log(1+\|z\|^2) = \partial\left(\frac{z}{1+z\bar z}\, d\bar z\right)$$
$$= \frac{\partial}{\partial z}\left(\frac{z}{1+z\bar z}\right) dz\wedge d\bar z$$
$$= \frac{1.(1+z\bar z) - z\bar z}{(1+z\bar z)^2}\, dz\wedge d\bar z.$$
$$= \frac{1}{(1+\|z\|^2)^2}\, dz\wedge d\bar z.$$

$$\omega_{FS} = \frac{i}{2}\partial\bar\partial\log(1+\|z\|^2) \qquad z = z_1/z_0$$
$$= \frac{i}{2}\partial\bar\partial\log\left(\frac{\|z_0\|^2+\|z_1\|^2}{\|z_0\|^2}\right)$$
$$= \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2) - \frac{i}{2}\underbrace{\partial\bar\partial\log\|z_0\|^2}_{"0}$$
$$= \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2)$$

Definition: The Fubini–Study form ω_{FS} on $\mathbb{CP}^n$ is defined as

$$\omega_{FS} \doteq \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2+\cdots+\|z_n\|^2).$$

Example: $n=2$, $\mathbb{CP}^2$, $\omega_{FS} = \frac{i}{2}\partial\bar\partial\log(\|z_0\|^2+\|z_1\|^2 + \|z_2\|^2)$

$$\Rightarrow \omega_{FS} = \frac{i}{2}\,\frac{(1+z_2\bar{z}_2)\,dz_1\wedge d\bar{z}_1 + (1+z_1\bar{z}_1)\,dz_2\wedge d\bar{z}_2}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^2}$$

$$+\frac{i}{2}\,\frac{z_1\bar{z}_2\,d\bar{z}_1\wedge dz_2 + z_2\bar{z}_1\,d\bar{z}_2\wedge dz_1}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^2}$$

$z_j = x_j + i y_j$, $j = 1,2$.

$$\omega_{FS}\wedge\omega_{FS} = 2\left(\frac{i}{2}\right)^2 \frac{dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2}{(1+z_1\bar{z}_1+z_2\bar{z}_2)^3}$$

$$= \frac{2}{(1+x_1^2+y_1^2+x_2^2+y_2^2)^3}\, dx_1\wedge dy_1\wedge dx_2\wedge dy_2$$

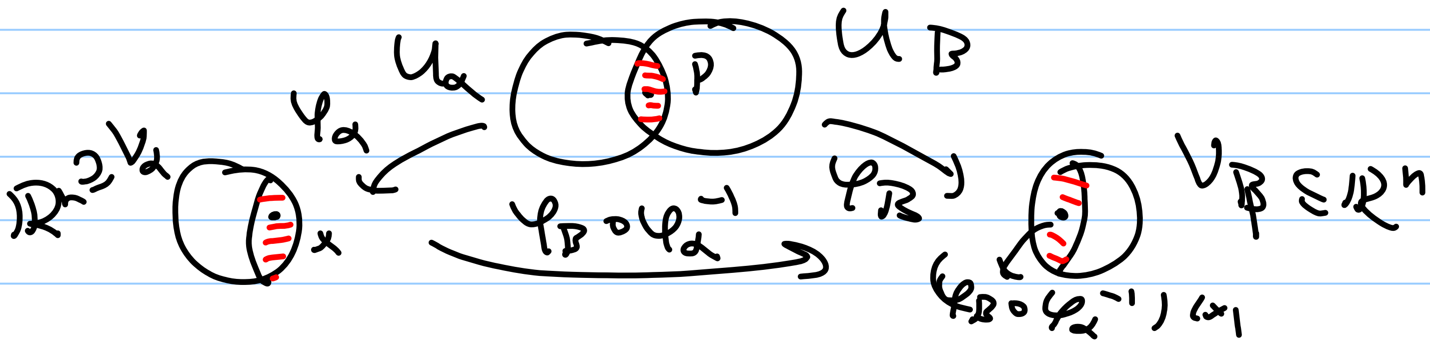
$$\int_{\mathbb{CP}^2}\omega_{FS}\wedge\omega_{FS} = \int_{\mathbb{R}^4}\omega_{FS}\wedge\omega_{FS} \overset{?}{=} \pi^2.$$

Vector Bundles:

M smooth manifold with an atlas $\{\varphi_\alpha : U_\alpha \to V_\alpha\}$.

$$M = \bigcup_\alpha U_\alpha = \dot{\bigcup_\alpha} V_\alpha \Big/ x \sim (\varphi_\beta \circ \varphi_\alpha^{-1})(x)$$

$U_\alpha \subseteq M$, $V_\alpha \subseteq \mathbb{R}^n$



$$T_x M = \bigcup_\alpha T_x U_\alpha = \dot{\bigcup_\alpha} T_x V_\alpha = \dot{\bigcup_\alpha} (V_\alpha \times \mathbb{R}^n) \Big/ (x, v) \sim \left((\varphi_\beta \circ \varphi_\alpha^{-1})(x), w\right)$$

when $w = D(\varphi_\beta \circ \varphi_\alpha^{-1})_{(x)}(v)$

$$T^* M = \bigcup_\alpha T^* U_\alpha = \dot{\bigcup_\alpha} T^* V_\alpha = \dot{\bigcup_\alpha} V_\alpha \times (\mathbb{R}^n)^* \Big/ \sim$$

$$(x, D(\varphi_\beta \circ \varphi_\alpha^{-1})^*_{(x)}(w)) \sim ((\varphi_\beta \circ \varphi_\alpha^{-1})(x), w)$$

Definition: let $P : E^{m+k} \to M^m$ be a smooth map of smooth manifolds satisfying the following conditions:

1) For any $p \in M$, $E_p = p^{-1}(p)$ is a k-dim'l real vector space.

2) There is an open cover $\{U_\alpha\}_{\alpha \in \Delta}$ of M so that

i) For each $\alpha \in \Delta$ there is a diffeomorphism

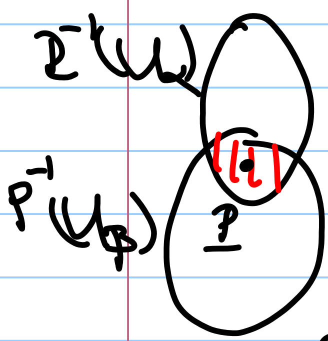
$$\phi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^k$$

ii) For each $\alpha \in \Delta$ and $p \in U_\alpha$ the restriction map $\phi_\alpha\big|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^k$ is a linear isomorphism of real vector spaces.

$E_p = p^{-1}(p)$

In this case, we say that $p : E^{m+k} \longrightarrow M^m$ is a smooth real vector bundle of rank k.

Transition Function:



$$\begin{tikzcd} & U_\alpha \times \mathbb{R}^k \arrow[d, "\phi_\beta \circ \phi_\alpha^{-1}"] & (p, v) \arrow[d, mapsto] \\ \cdot \arrow[ur, "\phi_\alpha"] \arrow[r, "\phi_\beta"'] & U_\beta \times \mathbb{R}^k & (p, (\phi_\beta \circ \phi_\alpha^{-1})(v)) \end{tikzcd}$$

$\phi_\beta \circ \phi_\alpha^{-1}\big|_p : \{p\} \times \mathbb{R}^k \longrightarrow \{p\} \times \mathbb{R}^k$ is a linear

Isomorphism.

$$U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{R})$$
$$p \longmapsto (\phi_\beta \circ \phi_\alpha^{-1}) : \{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$$

We write this as

$$\phi_\beta \phi_\alpha^{-1}(p, v) = (p, \psi_{\beta\alpha}(p)(v)), \quad \psi_{\beta\alpha}(p) \in GL(k, \mathbb{R})$$

$\psi_{\beta\alpha}$ is called a transition function for the vector bundle. Note that they satisfy the cocycle condition $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$.

Ex $T_x \mathbb{C}P^1 \cong U_0 \times \mathbb{C} \,\dot{\cup}\, U_1 \times \mathbb{C} \big/ (z, v) \sim (1/z, -\frac{1}{z^2} v)$

$$\phi_0 : p^{-1}(U_0) \longrightarrow U_0 \times \mathbb{C} \qquad (z, v)$$
$$\phi_1 : p^{-1}(U_1) \longrightarrow U_1 \times \mathbb{C} \qquad (1/z, -\tfrac{1}{z^2} v)$$

$$\psi_{01} : U_0 \cap U_1 \longrightarrow GL(1, \mathbb{C})$$
$$z \longmapsto \left[-\frac{1}{z^2}\right]$$

This examples shows that the above definition can be made for the field $\mathbb{C}$ or even $\mathbb{H}$.

Over $\mathbb{C}$ we get complex vector bundles and over $\mathbb{H}$ we get Quaternion vector bundles.

<u>Remark:</u> If M is a smooth manifold and $\{U_\alpha\}_{\alpha\in\Delta}$ is an open cover. Let

$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k,\mathbb{F})$ $(\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H})$

be smooth functions satisfying the conditions

i) $\psi_{\alpha\beta}(p) = (\psi_{\beta\alpha}(p))^{-1}$

ii) $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$, for all α, β, γ, then we obtain a smooth $\mathbb{F}$-vector bundle of rank k as follows:

$$\begin{array}{ll} E = \dot{\bigcup}_{\alpha} \left(U_\alpha \times \mathbb{F}^k\right) \big/ (p,v) \sim (p, \psi_{\alpha\beta}(p)(v)) \\ \downarrow \\ M & \text{if } p \in U_\alpha \cap U_\beta . \end{array}$$

<u>Operations on Vector Bundles:</u>

<u>Summation:</u> Let $E_i \longrightarrow M$, $i = 1, 2$, be vector

bundles of rank k_1 and k_2. Suppose $E_i \to M$ has transition functions $\{\psi^i_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k_i, \mathbb{F})\}$

Then the direct sum $E_1 \oplus E_2 \longrightarrow M$ of E_1 and E_2 is the vector bundle with transition functions

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k_1+k_2, \mathbb{F})$$

$$p \longmapsto \left[\begin{array}{c|c} \psi^1_{\alpha\beta} & 0 \\ \hline 0 & \psi^2_{\alpha\beta} \end{array}\right]_{(k_1+k_2)\times(k_1+k_2)}$$

so that the direct sum is a vector bundle of rank k_1+k_2.

<u>Determinant Line Bundle:</u>

Let $p: E^{n+k} \to M^n$ be a vector bundle of rank k ($\mathbb{F} = \mathbb{R}, \mathbb{C}$), with transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(k, \mathbb{F})$.

Consider the composition

$$\det \circ \psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{K}^* = GL(1, \mathbb{F})$$

$\varphi_{\alpha\beta} = \det \circ \psi_{\alpha\beta}$ still satisfy the conditions:

$$\varphi_{\alpha\beta} = \det(\Psi_{\alpha\beta})$$

i) $$\varphi_{\beta\alpha} = \det(\Psi_{\beta\alpha}) = \det(\Psi_{\alpha\beta}^{-1})$$
$$= \left(\det(\Psi_{\alpha\beta})\right)^{-1} = (\varphi_{\alpha\beta})^{-1}$$

ii) $$\varphi_{\alpha\beta}\circ\varphi_{\beta\gamma} = \det(\Psi_{\alpha\beta})\cdot\det(\Psi_{\beta\gamma})$$
$$= \det(\Psi_{\alpha\beta}\cdot\Psi_{\beta\gamma})$$
$$= \det(\Psi_{\alpha\gamma})$$
$$= \varphi_{\alpha\gamma}.$$

The rank 1 vector bundle with transition functions $\varphi_{\alpha\beta} = \det(\Psi_{\alpha\beta})$ is called the determinant line bundle of $E \to M$ and denoted by $\det(E) \to M$.