Geometry I

M A T H 3 7 3

**FINAL EXAMINATION**

(*Duration : 110 mins.*)

9th January 1999

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**1.**

Find the image of $y = x$ under the following transformations :

(a) $Hom((-1,0),6)$ .

(b) $Rot((0,-2),\pi/6)$ .

(c) $Inv((0,10\sqrt{2}),\sqrt{3200})$ .

(a) $y = x - 5$

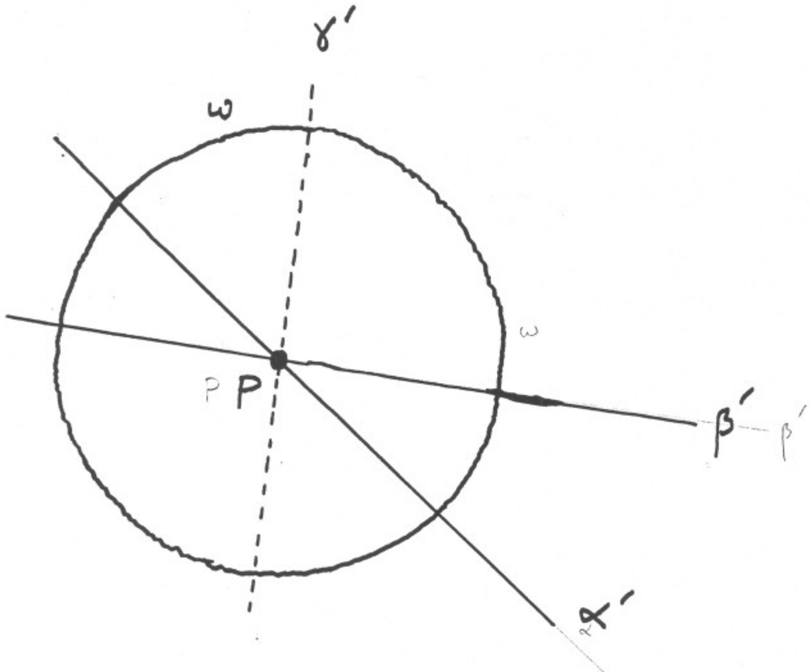
(b) $y - \sqrt{3} + 2 = \frac{\sqrt{3}+1}{\sqrt{3}-1}(x+1)$

(c) $(x-2)^2 + (y - 10\sqrt{2} + 2)^2 = 8$

**2.**

Consider circles $\alpha, \beta$ with $\alpha \cap \beta = \{P, Q\}$. Let $\omega$ be a circle which is perpendicular to both $\alpha$ and $\beta$. Prove that any circle through $P$, $Q$ is perpendicular to $\omega$.

Under an inversion with center $P$, the circles $\alpha, \beta$ are transformed into straight lines $\alpha', \beta'$ through $P$. The image $\omega'$ of $\omega$ is a circle perpendicular to




$\alpha'$ and $\beta'$. Hence $P$ is the center of $\omega'$. Now, for any circle $\gamma$ through $P, Q$, the image $\gamma'$ of $\gamma$ is a straight line through $P$. $P$ being the center of $\omega'$, $\gamma'$ is perpendicular to $\omega'$. Consequently $\gamma$ is perpendicular to $\omega$.

3.

(a) Given a triangle $UVW$, consider $X \in VW - \{V, W\}$, $Y \in WU - \{W, U\}$, $Z \in UV - \{U, V\}$. Prove that

$$\frac{XV}{XW} \cdot \frac{YW}{YU} \cdot \frac{ZU}{ZV} = \frac{\sin(< XUV)}{\sin(< XUW)} \cdot \frac{\sin(< YVW)}{\sin(< YVU)} \cdot \frac{\sin(< ZWU)}{\sin(< ZWU)} .$$

(b) Consider a positively oriented triangle $ABC$. Let $BB''C'C$, $CC''A'A$, $AA''B'B$ be positively oriented squares (constructed "outside" on the respective sides $BC$, $CA$, $AB$ of the triangle $ABC$). Let $C''A'$ and $A''B'$, $A''B'$ and $B''C'$, $B''C'$ and $C''A'$ intersect in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ respectively. Prove that $A\tilde{A}$, $B\tilde{B}$, $C\tilde{C}$ are concurrent.

(a) Start with

directed!

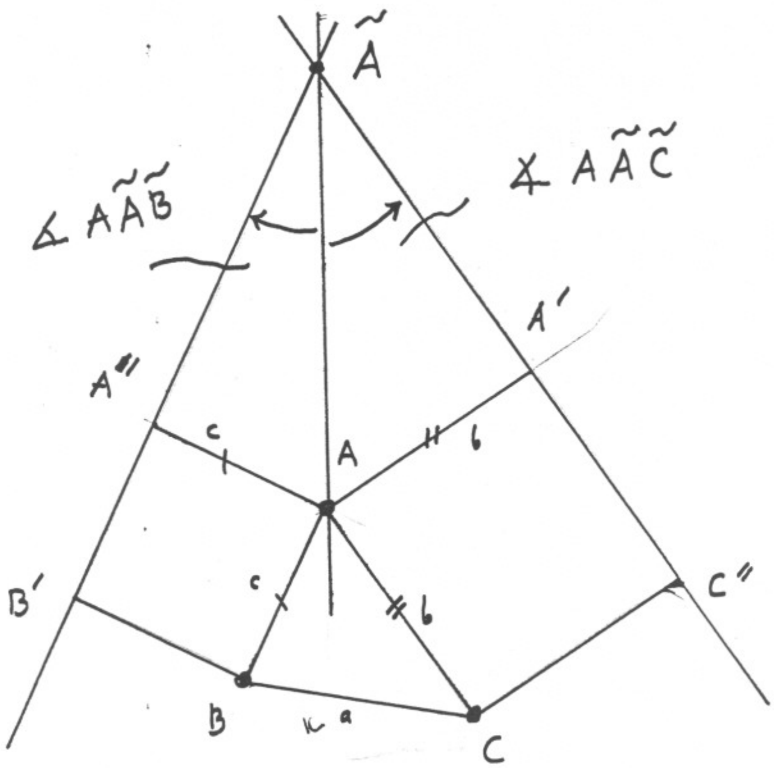
$$\frac{XV}{XW} = \frac{\text{area}(XVU)}{\text{area}(XWU)} = \frac{\frac{1}{2}|UV||UX|\sin(\measuredangle XUV)}{\frac{1}{2}|UX||UW|\sin(\measuredangle XUW)} = \frac{|UV|\sin(\measuredangle XUV)}{|UW|\sin(\measuredangle XUW)}$$

and proceed in cyclic manner and multiply out.

(b) (1) TRIVIAL SOLUTION I HAVE MISSED:

Desargues' Theorem!

(2)




Observe that $\dfrac{\sin(\measuredangle A\tilde{A}\tilde{B})}{\sin(\measuredangle A\tilde{A}\tilde{C})} = -\dfrac{c}{b}$.

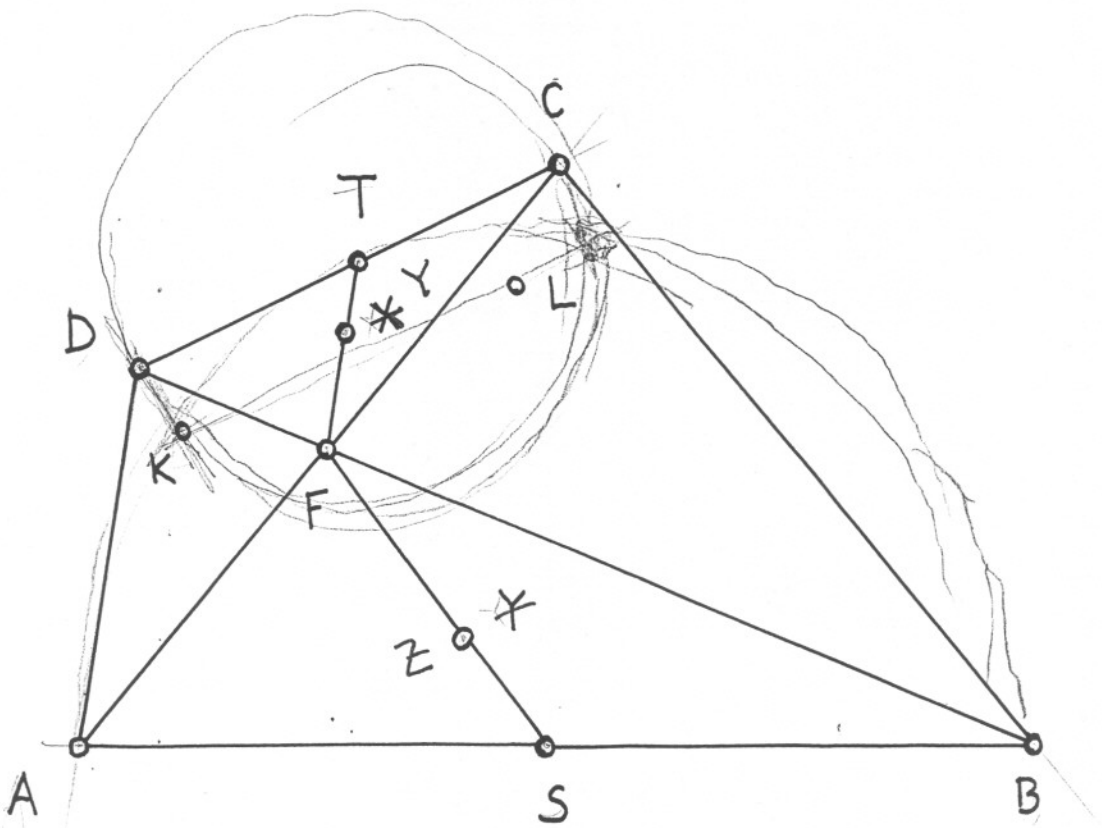
Proceed in cyclic manner, multiply out, apply (a).

4.

(a) Given a triangle $UVW$ and points $P \in UW$, $Q \in UV$, let $\beta, \gamma$ be circles with respective diameters $[V, P]$, $[W, Q]$. Prove that the orthocenter of $UVW$ lies on the radical axis of $\beta$ and $\gamma$.

(b) Consider a quadrangle $ABCD$ with $AC \cap BD = \{F\}$. Let $K, L$ be the orthocenters of $AFD$, $BFC$, let $S, T$ be the midpoints of $[A, B]$, $[C, D]$ respectively. Prove that $KL \perp ST$. (*Hint* : Circles of diameters $[A, B]$,$[C, D]$.)

(c) Let $Y, Z$ be the respective centroids of $CFD$, $AFB$. prove that $KL \perp YZ$.




(a) My lectures.

(b) Let $\alpha, \beta$ be the circles of diameters $[A,B]$, $[C,D]$ respectively. Applying (a) in AFD it is seen that the radical axis of $\alpha$ and $\beta$ passes through $K$. Similar argument with $L$: $KL$ <u>is</u> the radical axis of $\alpha$ and $\beta$; as such it is perpendicular to the line $ST$ which joins the centers of $\alpha$ and $\beta$.

(c) Since $\frac{Y\cancel{X}T}{Y\cancel{X}F} = \frac{Z\cancel{Y}S}{Z\cancel{X}F} = -\frac{1}{2}$, $\overset{YZ}{\cancel{XY}} \parallel ST \perp KL$.