p1 ①

Geometry I

MATH 373

FIRST MIDTERM

(*Duration : 110 mins.*)

5th December 1998

[5 + (10 + 5)+ 15] , [5 + 5 + 10 + 10] , [5 + 10 + 20]

1.

Let Δ denote the area of the triangle ABC.

(a) Prove that

$$\Delta = sr = (s-a)r_a.$$

(b) If $r + r_b + r_c = r_a$, prove that $A = \pi/2$. Is the converse true ?

(c) Find A if $3r + r_b + r_c = 3r_a$.

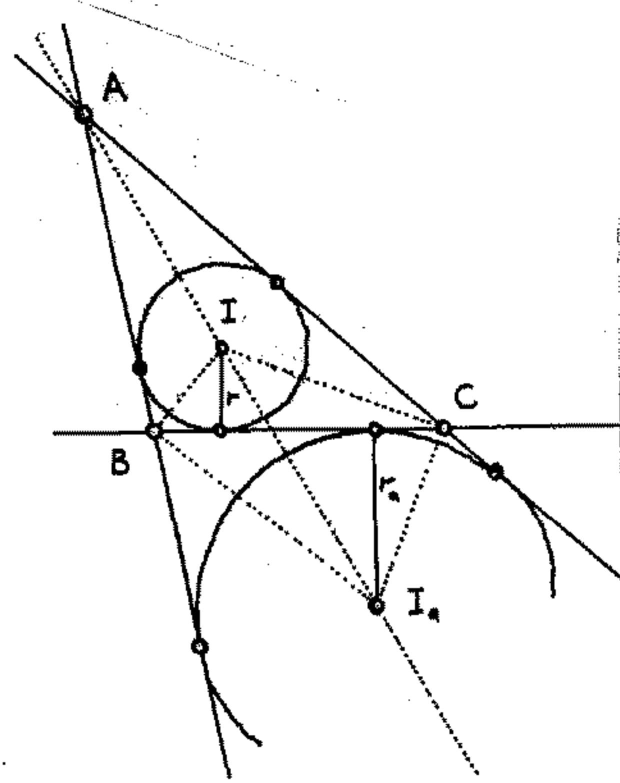
a) $\Delta = \text{Area}(IBC) + \text{Area}(ICA) + \text{Area}(IAB)$

$= \frac{1}{2} r \cdot a + \frac{1}{2} r \cdot b + \frac{1}{2} r c = sr$

$= \text{Area}(I_a CB) + \text{Area}(I_a CA) + \text{Area}(I_a AB)$

$= -\frac{1}{2} r_a a + \frac{1}{2} r_a b + \frac{1}{2} r_a c$

$= (s-a) r_a$



(b) $r + r_b + r_c = r_a$ gives $\Delta\left(\frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c}\right) = \Delta \frac{1}{s-a}$

from which one obtains

$$s(s-a) = (s-b)(s-c)$$

that can be regrouped to result in $a^2 = b^2 + c^2$. Hence $A = \pi/2$. The converse holds as the above numerical statements are equivalent.

(c) Similarly $3r + r_b + r_c = 3r_a$ results in

$$s(s-a) = 3(s-b)(s-c)$$

and

$$a^2 = b^2 + c^2 - 2bc \cdot \tfrac{1}{2}$$

which implies $A = \pi/3$.

2.

(a) In the triangle ABC prove that I is the orthocenter of $I_aI_bI_c$.

(b) If ABC is not eqilateral prove that OI passes through the circumcenter of $I_aI_bI_c$.

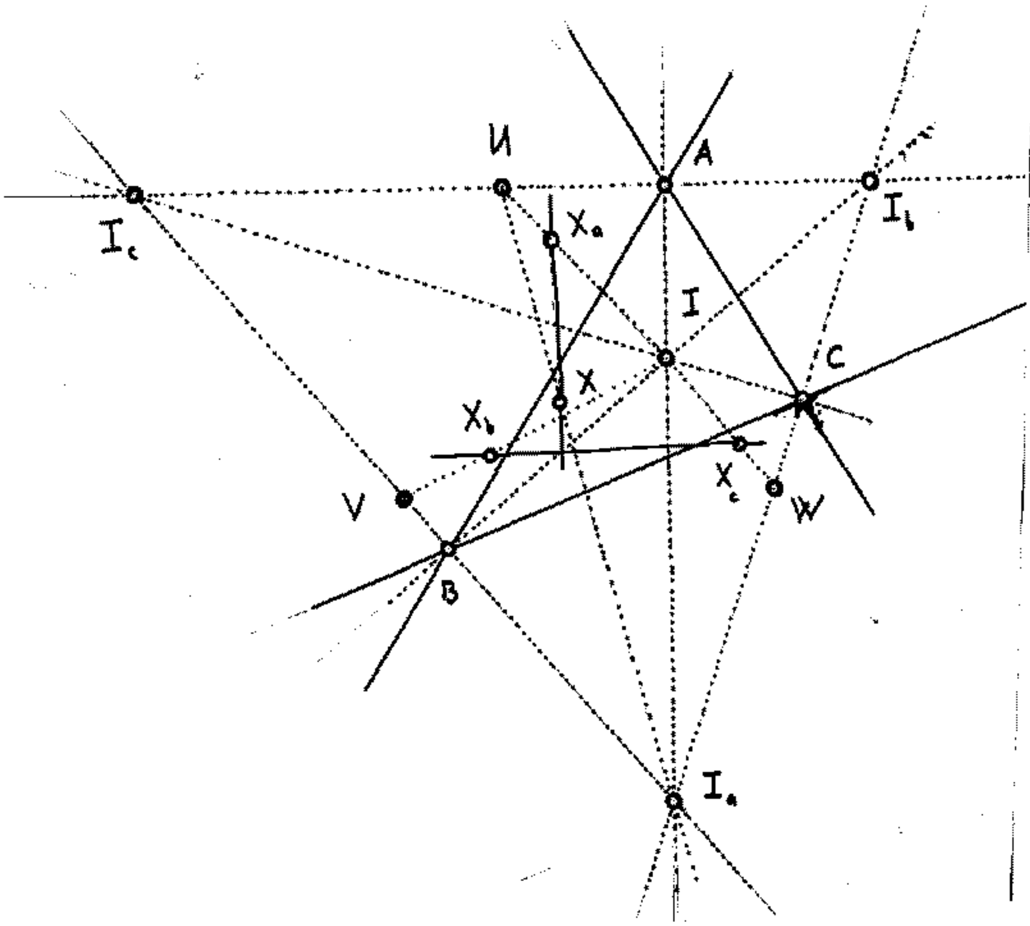
(c) Prove that (O) passes through the midpoint of the line segment $[I_b, I_c]$.

(d) If X, X_a, X_b, X_c are the respective centroids of $I_aI_bI_c$, II_bI_c, I_aII_c, I_aI_bI, prove that X is the orthocentre of $X_aX_bX_c$.

(a) II_a, I_bI_c being internal and external bisectors of A respectively, they are perpendicular to one another. Similarly $II_b \perp I_cI_a$, $II_c \perp I_aI_b$. Consequently I is the orthocentre of $I_aI_bI_c$.

(b) (O) is the 9-point circle of $I_aI_bI_c$ whereof I is the orthocentre. Therefore OI is the Euler line in $I_aI_bI_c$ and contains the circumcentre of $I_aI_bI_c$.

(c) (O) being the 9-point circle of $I_aI_bI_c$, it bisects the sides of the triangle in question.



d) Let U, V, W be midpoints of $[I_b, I_c]$, $[I_c, I_a]$, $[I_a, I_b]$.

$XU : XI_a = X_aU : X_aI = -1:2$ hence $XX_a \parallel II_a$.

Similarly $X_bV : X_bI = X_cW : X_cI = -1:2$ hence $X_bX_c \parallel VW \parallel I_bI_c$. Thus

$$XX_a \perp X_bX_c$$

similarly

$$XX_b \perp X_cX_a$$

$$XX_c \perp X_aX_b.$$

which show that X is the orthocentre of $X_aX_bX_c$.

3.

(a) In the triangle ABC, let AH, BH, CH meet BC, CA, AB in D, E, F respectively. Prove that

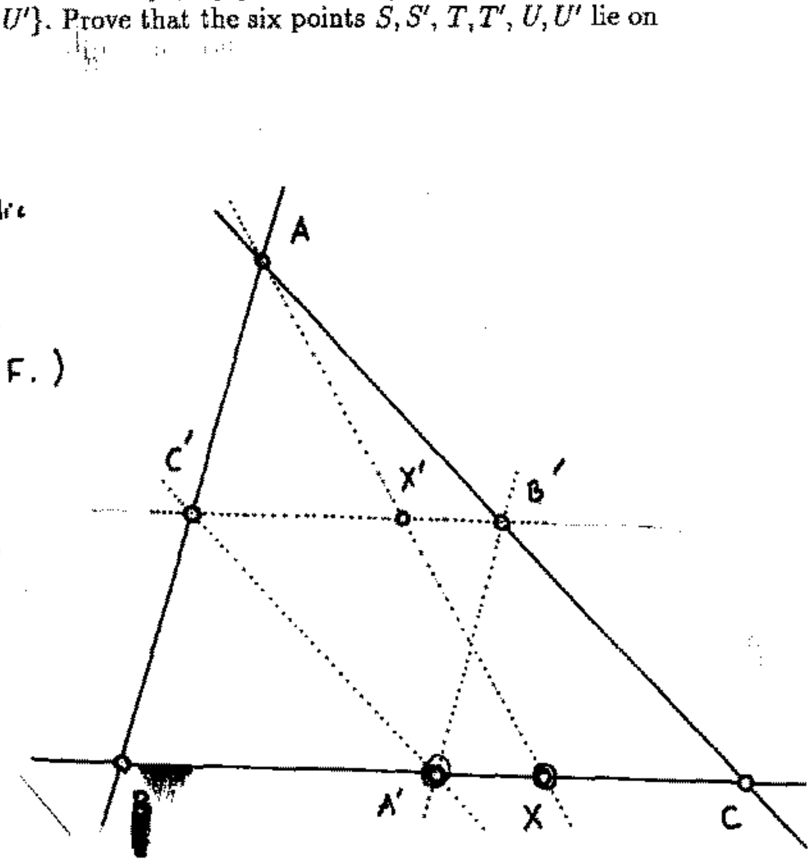
$$HA \cdot HD = HB \cdot HE = HC \cdot HF.$$

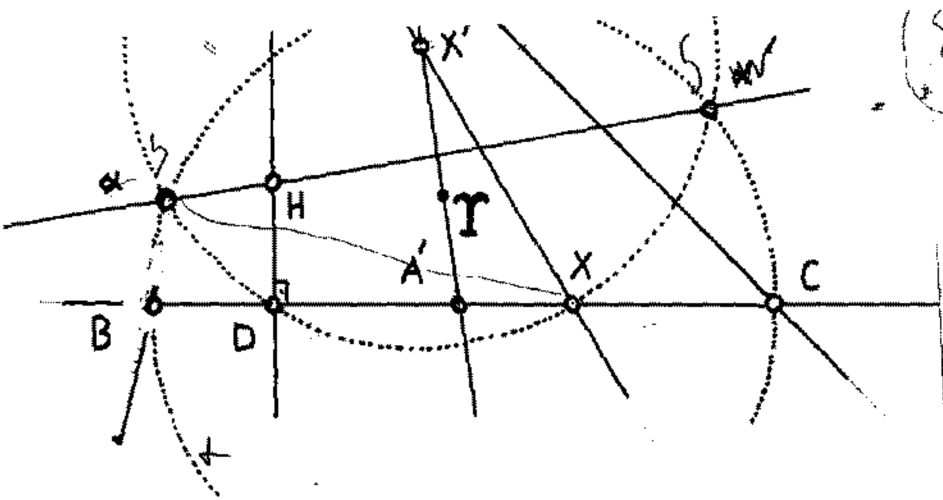
(b) Let A', B', C' be midpoints of $[B, C]$, $[C, A]$, $[A, B]$ respectively. Consider $X \in BC - \{B, C\}, Y \in CA - \{C, A\}, Z \in AB - \{A, B\}$ such that AX, BY, CZ are concurrent. If X', Y', Z' are midpoints of $[A, X]$, $[B, Y]$, $[C, Z]$ respectively, prove that $X'A'$, $Y'B'$, $Z'C'$ concur in a point Υ.

(c) Let α, β, γ be the respective circles of diameter $[B, C]$, $[C, A]$, $[A, B]$. Let α', β', γ' be the respective circles of diameter $[A, X]$, $[B, Y]$, $[C, Z]$. Let $\alpha \cap \alpha' = \{S, S'\}$, $\beta \cap \beta' = \{T, T'\}$, $\gamma \cap \gamma' = \{U, U'\}$. Prove that the six points S, S', T, T', U, U' lie on a circle of center Υ.

(a) B, C, E, F are concyclic hence HB · HE = HC · HF (the power of H w.r.to the circle containing B, C, E, F.)

(b) $\dfrac{X'B'}{X'C'} = \dfrac{XC}{XB}$ similarly $\dfrac{Y'C'}{Y'A'}$ and $\dfrac{Z'A'}{Z'B'}$. Apply Ceva in $A'B'C'$.





S, T, T', V, U'

SS' ~~$\alpha\alpha'$~~ is the radical axis of the circles with diameter $[B,C]$ and $[A,X]$ ~~consequently~~ $H \in \alpha\alpha'$. Similarly $H \in \cancel{\beta\beta'}$ TT', $\cancel{\gamma\gamma'}$ WW'.

Therefore

$$\cancel{H\alpha}\, S \cdot \cancel{H\alpha'}\, S' = HD \cdot HA = HE \cdot HB = \cancel{H\beta}\, T \cdot \cancel{H\beta'}\, T'.$$

Consequently $\alpha, \alpha', \beta, \beta'$ lie on a circle which has center (!) Υ.