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Geometry I

MATH 373

SECOND MIDTERM

(Duration : 110 mins.)

26th December 1998

[5 + 10 + 10 + 10] , [10 + 10 + 15] , [10] , [10 + 5 + 5]

1.

(a) What is the image of the line $2x + y = 2$ under Tr_u where $u = [-1, 3]$?

(b) What is the image of the line $x = 4y - 2$ under $Hom((1, -1), 3)$?

(c) What is the image of the circle $x^2 + y^2 - 6y + 5 = 0$ under Ref_k where k is the line $x = \sqrt{3}(y - 1)$?

(d) What is the image of the line $y = x$ under $Rot((0, -1), \pi/6)$?

(a) Take any point, say $(0,2)$ on the line. It is mapped into $(-1,5)$. Consequently the image is a line of the form $2x + y = m$ containing $(-1,5)$. Therefore it is $2x + y = 3$.

(b) Similarly take any point, say $(2,1)$ on the line. It is mapped into $(4,5)$. Consequently the image is a line of the form $x = 4y - n$ containing $(4,5)$. Therefore it is $x = 4y - 16$.

(c) The center of the circle in question is $(0,3)$. Its radius is 2. A simple inspection reveals that the reflection in $x = \sqrt{3}(y-1)$ sends $(0,3)$ into $(\sqrt{3}, 0)$. The image is the circle $(x - \sqrt{3})^2 + y^2 = 4$

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(d) The new line has slope $\tan(\pi/4 + \pi/6) = \dfrac{1+\sqrt{3}}{\sqrt{3}-1}$

and contains the point

$$\mathrm{Rot}\left((0,-1), \pi/6\right)(0,0) = \left(-\tfrac{1}{2}, -\left(1 - \tfrac{\sqrt{3}}{2}\right)\right).$$

Consequently its equation is

$$y + 1 - \frac{\sqrt{3}}{2} = \frac{\sqrt{3}+1}{\sqrt{3}-1}\left(x + \frac{1}{2}\right).$$

2.

Let A, B, C, D and $O \neq A, B, C, D$ be points on a line k, A', B', C', D' be the respective images of A, B, C, D with respect to an inversion of centre O. Prove that

$$\frac{CA}{CB} \div \frac{DA}{DB} = \frac{C'A'}{C'B'} \div \frac{D'A'}{D'B'}.$$

(*Hint* : Identify k with the x-axis and choose O as the origin !)

Identifying k with the x-axis and O with the origin, let A, B, C, D have coordinates a, b, c, d respectively. If the power of the inversion is k, the coordinates of A', B', C', D' are respectively k/a, k/b, k/c, k/d. Consequently

$$\frac{C'A'}{C'B'} : \frac{D'A'}{D'B'} = \frac{k/c - k/a}{k/c - k/b} : \frac{k/d - k/a}{k/d - k/b} = \frac{-k\,\frac{c-a}{ca}}{-k\,\frac{c-b}{cb}} : \frac{-k\,\frac{d-a}{da}}{-k\,\frac{d-b}{db}}$$

$$= \frac{c-a}{c-b} : \frac{d-a}{d-b} = \frac{CA}{CB} : \frac{DA}{DB}$$

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Consider a convex positively oriented quadrangle $ABCD$ with positively oriented equilateral triangles BAP, CBQ, DCR, ADS, constructed on its sides (i. e. on the "outside"). Let

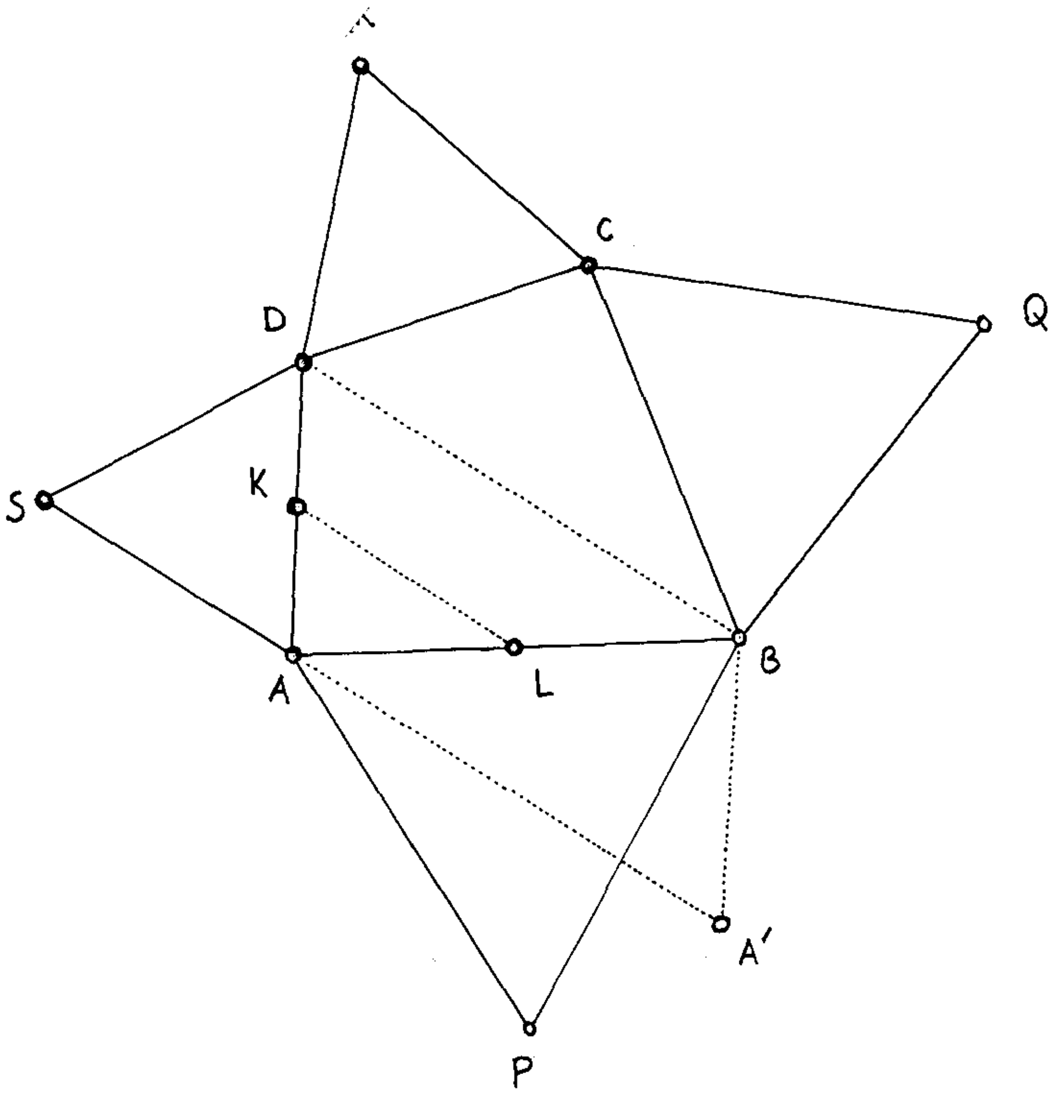
$$\alpha = Rot(P, \frac{\pi}{3}) \circ Rot(Q, \frac{\pi}{3}) \circ Rot(R, \frac{\pi}{3})$$

$$\beta = Rot(Q, \frac{\pi}{3}) \circ Rot(R, \frac{\pi}{3}) \circ Rot(S, \frac{\pi}{3})$$

(a) Prove that $\alpha^2 = Id$, $\beta^2 = Id$.

(b) Quite generally, what is the composition of two half-turns ?

(c) Let A' be the image of A under $\beta \circ \alpha$. What can you say about the quadrangle $AA'BD$?



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(a) α, being the composition of three rotations each through $\pi/3$, a half-turn; therefore it is involutive. Similarly β.

(b) $Rot(Q,\pi) \circ Rot(P,\pi) = Tr_{2\overrightarrow{PQ}}$

(c) Clearly $\alpha = Rot(K,\pi)$, $\beta = Rot(L,\pi)$, where K, L are the respective midpoints of $[D,A]$, $[A,B]$. Consequently

$$\beta \circ \alpha = Tr_{2\overrightarrow{KL}} = Tr_{\overrightarrow{DB}}$$

Therefore $AA'BD$ is a parallelogram.

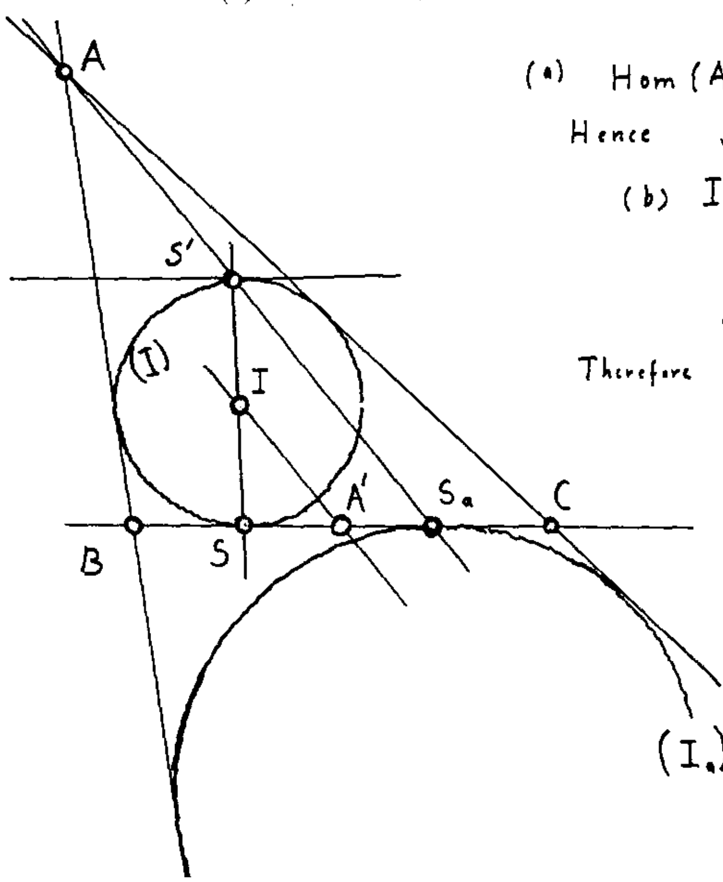
4.

Let (I), (I_a) touch BC in S, S_a, respectively. Let $[SS']$ be a diameter of (I).

(a) Prove that A, S', S_a] are collinear.

(b) Let A', B', C', be the midpoints of $[B,C]$, $[C,A]$ $[A,B]$. Prove that I is the Nagel point of $A'B'C'$.

(c) Prove that I, G, N are collinear and $GI : GN = -1 : 2$.



(a) $Hom(A, \frac{r}{r_a})$ sends S_a into S' (why?) Hence S', S_a, A are collinear.

(b) I, A' are midpoints of $[S,S']$, $[S,S_a]$. Hence $A'I \parallel S_aS = AS_a$. AS_a, BS_b, CS_c meet in N. ~~Apply~~ Therefore "medial homothety" sends N into the point of intersection of $A'I$, $B'I$, $C'I$!

(c) Now obvious using the "medial homothety" !