Geometry I

MATH 373

FINAL EXAMINATION

(*Duration : 120 mins.*)

19th January 2000

[5 + 5 + 5 + 5 + 5] , [10 + 10] , [10 + 15], [10 + 10 + 10]

1.

Let ABC be a positively oriented isosceles triangle with $|AB| = |AC|, < BAC = 2\theta$. Let S be the midpoint of $[B, C]$, P be the foot of the perpendicular from S on AB, M be the midpoint of $[P, S]$. Consider the transformation

$$f = Hom(B, \lambda) \circ Tr_u \circ Rot(S, -\pi/2)$$

where $\lambda = 2\tan\theta$, $u = -\overrightarrow{BS}$

(A) Prove that $f(S) = B$.

(B) Prove that $f(A) = C$.

(C) Prove that P is the midpoint of $[B, f(P)]$.

(D) What can you say about $f(M)$?

(E) Prove that $AM \perp PC$.

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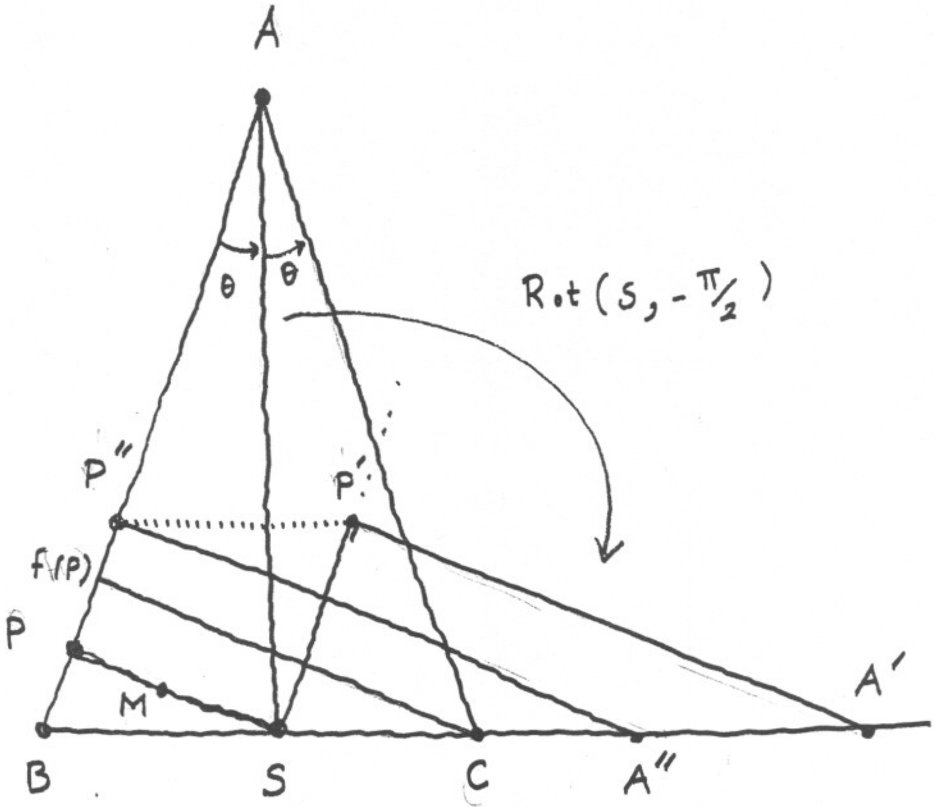


Assume w.l.o.g. that $|SA| = 1$. As a consequence $|BC| = 2\tan\theta$.

[redacted]

(A) $S \xrightarrow{Rot(S,-\pi/2)} S \xrightarrow[u = -\overrightarrow{BS}]{Tr_u} B \xrightarrow{Hom(B,\lambda)} B$

(B) $Rot(S, -\pi/2)$ sends A into a point $A' \in BC$, on the same side of B as C. $A'' = Tr_u(A')$ has the same property and $|BA''| =$ [redacted] $|SA| = 1$.

Consequently [redacted] $|Bf(A)| \overset{2}{=} \tan\theta$ and

$f(A) = C$.

(C) Let $p' = Rot(S, -\pi/2)(P)$, $P'' = Tr_u(p')$. Note that $AB = AP \perp A'p' \parallel A''p'' \parallel Cf(P)$. As S is midpoint of $[B, C]$, P has to be the midpoint of $[B, f(P)]$.

(D) M is the midpoint of $[S, P]$. Consequently $f(M)$

is the midpoint of $[B, f(P)]$ which is P. Therefore

$f(M) = P$

(E) [redacted]

f(A $PC = f(M)f(A) \parallel Tr_u \circ Rot(s, -\pi/2)(AM)$

$\parallel Rot(s, -\pi/2)(AM) \perp AM$

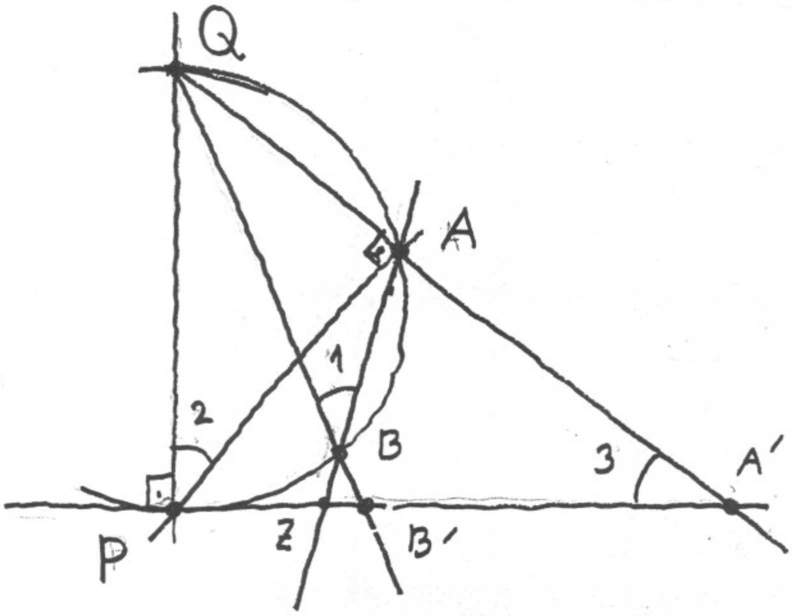
2.

Consider a circle γ with diameter $[P, Q]$. Let $A, B \in \gamma - \{P, Q\}$ be distinct points, t be the tangent to γ at P. Suppose that QA, QB intersect t in A', B' respectively.

(A) Prove that A, B, A', B' are concyclic.

(B) Let AB intersect t in Z. Prove that

$$|ZP|^2 = ZA'.ZB' \quad .$$



(A) Follow 1, 2, 3 !

(B) $|ZP|^2$ = power of Z w.r.to circle of diameter $[P, Q]$

$= ZA \cdot ZB$

= power of Z w.r.to the circle A, B, A', B'.

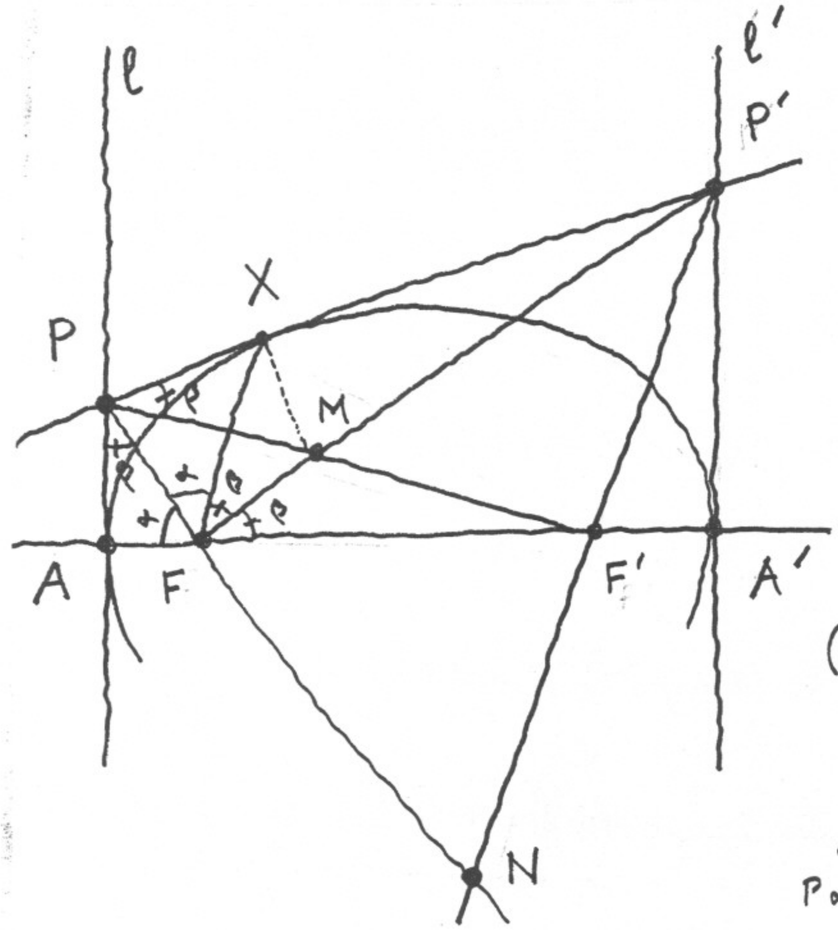
$= ZA'.ZB'$

3.

Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ at A, A'. For ~~any~~ the tangent t to φ at $X \in \varphi$ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$.

(A) Prove that $<(FP, FP') = \pi/2$

(B) Prove that $FP, F'P'$ and the normal to φ at X are concurrent.



(A) By Poncelet II

$\alpha = \measuredangle AFP = \measuredangle PFX$

$\beta = \measuredangle XFP' = \measuredangle P'FA'$

Since $2\alpha + 2\beta = \pi$, we have

$\measuredangle PFP' = \alpha + \beta = \pi/2$.

Therefore $FP \perp FP'$.

(Similarly $F'P \perp F'P'$)

(B) By Poncelet I

$\measuredangle MPX = \measuredangle APF$

$= \measuredangle A'FP'$ (Again Poncelet I)

$= \measuredangle P'FX$

which shows that P, F, X, M are concyclic and $XM \perp PP'$ in particular, where M is the point in which FP' and $F'P$ intersect. Obviously M is the orthocenter of NPP'; XM is the altitude of NPP' through N.

4.

In an acute angled triangle ABC with orthocenter H and circumradius R, let AH intersect BC in $\tilde{A}$.

(A) Prove that $|AH| = 2R\cos A$.

(B) Prove that $|\tilde{A}H| = 2R\cos B\cos C$.

(C) Let B', C' be the midpoints of $[C,A]$, $[A,B]$ respectively. Prove that H lies on $B'C'$ iff

$$\tan B\tan C = 2 \ .$$

(A), (B) $\rightsquigarrow$ standard.

(C) If $H \in B'C'$ then H is the midpoint of $[A\tilde{A}]$ and

$$|AH| = |\tilde{A}H|$$

Thus

$$\cos A = \cos B\cos C$$

$$-\cos(B+C) = \cos B\cos C$$

$$\sin B\sin C = 2\cos B\cos C$$

$$\tan B\tan C = 2 \quad ✓$$