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Geometry I

MATH 373

FIRST MIDTERM

(*Duration : 100 mins.*)

9th November 1999

[30] , [10 + 10 + 20] , [30]

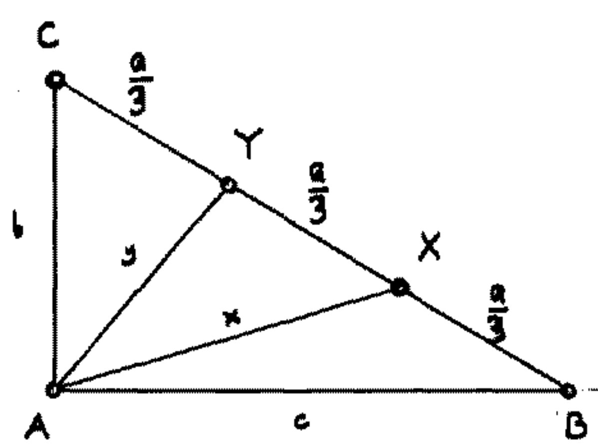
1.

Let $A = \pi/2$ in ABC. Consider $X, Y, \in BC$ such that

$$\frac{XB}{XC} = -\frac{1}{2}, \ \frac{YB}{YC} = -2.$$

Put $x = |AX|$, $y = |AY|$ and prove that

$$x^2 + y^2 = \frac{5}{9}a^2$$



Remember Stewart;

" $ax^2 = pb^2 + qc^2 - pqa$ "

$$ax^2 = \frac{a}{3}b^2 + \frac{2a}{3}c^2 - \frac{a}{3}\frac{2a}{3}a$$

$$ay^2 = \frac{2a}{3}b^2 + \frac{a}{3}c^2 - \frac{2a}{3}\frac{a}{3}a$$

Adding these and cancelling a,

$$x^2 + y^2 = b^2 + c^2 - \frac{4a^2}{9} = \frac{5a^2}{9}$$

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2.

Let Δ denote the area of the triangle ABC. Prove that

(a) $\Delta = sr = (s-a)r_a$.

(b) $\Delta = \dfrac{abc}{4R}$.

(c) $r_a + r_b + r_c = 4R + r$

(a), (b) $\longrightarrow$ Lectures ...

(c)
$$r_a + r_b + r_c = \Delta\left[\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right]$$

$$= \Delta\left[\frac{3s^2 - 2s(a+b+c) + bc+ca+ab}{(s-a)(s-b)(s-c)}\right]$$

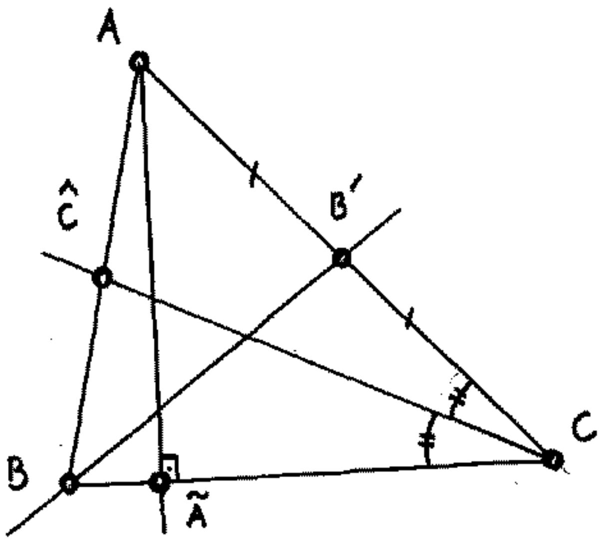
$$= \frac{\Delta}{s}\left[\frac{s^3 - (a+b+c)s^2 + (bc+ca+ab)s}{(s-a)(s-b)(s-c)}\right]$$

$$= \frac{\Delta}{s}\left[1 + \frac{abc}{(s-a)(s-b)(s-c)}\right]$$

$$= r + \frac{\Delta\, abc}{\Delta^2} = r + 4R$$

In the triangle ABC, the altitude through A, the median through B and the internal angle bisector through C are concurrent. Prove that

$$\tan C = \frac{\sin A}{\cos B}.$$



If $A\tilde{A}$, BB', $C\hat{C}$ are concurrent, then

$$-1 = \frac{\tilde{A}B}{\tilde{A}C} \cdot \frac{B'C}{B'A} \cdot \frac{\hat{C}A}{\hat{C}B} \overset{!}{=} \left(-\frac{c\cos B}{b\cos C}\right)(-1)\left(-\frac{b}{a}\right)$$

$$\Downarrow$$

$$\tan C = \frac{\sin A}{\cos B} \quad \text{(in particular } B \neq \pi/2 \text{ !)}$$