p17

Geometry I

MATH 373

SECOND MIDTERM

(*Duration : 100 mins.*)

24th December 1999

[7 + 8 + 10] , [15 + 10] , [15 + 15 + (10 + 10)]

1.

(a) What is the image of the line $3x - y - 4 = 0$ under $T_{[1,-3]}$?

(b) What is the image of the circle $x^2 + y^2 - 5y + 1 = 0$ under Ref_k , where k is the line $y = 2x$?

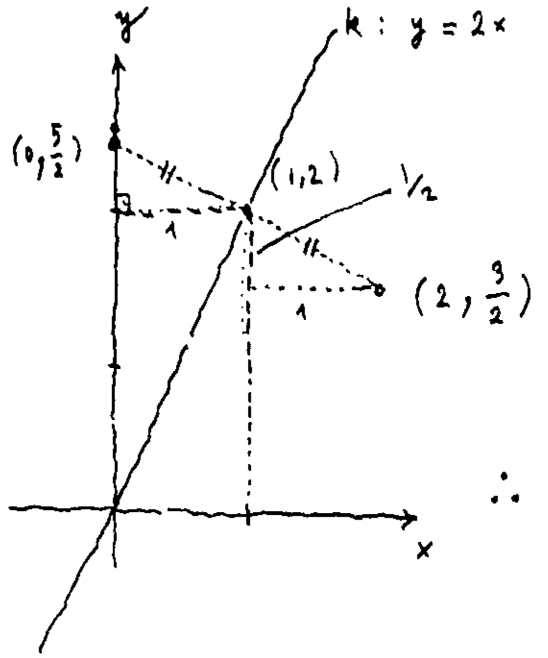
(c) What is the image of the line $x = 1$ under $Rot((-1, 0), \pi/3)$?

(a) The line in question contains the point $(0, -4)$ which moves into $(1, -7)$ under $T_{[1,-3]}$. The line required is

$$3x - y = 3.1 - (-7) = 10$$

$$3x - y - 10 = 0$$

(b)
The circle in question is the circle of center $(0, \frac{5}{2})$, and radius $\frac{\sqrt{21}}{2}$.



By a simple inspection it is found that

$$Ref_k((0, 5/2)) = (2, \tfrac{3}{2})$$

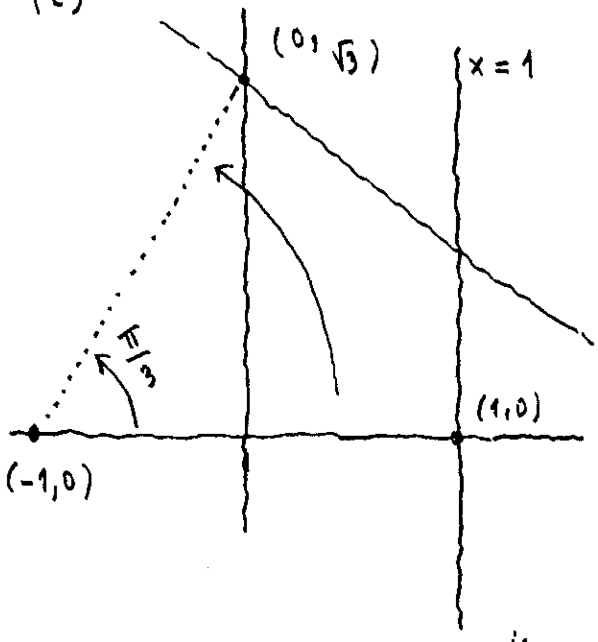
$\therefore$ The image of the circle is

$$\boxed{(x-2)^2 + (y - \tfrac{3}{2})^2 = \frac{21}{4}}$$

or :

$$\boxed{x^2 + y^2 - 4x - 3y + 1 = 0}$$

(c)



The point (1,0) moves into $(0, \sqrt{3})$ under $Rot((-1,0), \pi/3)$.

Under the same rotation $x = 1$ is turned into a line of the form

$$x + \sqrt{3}\, y = m$$

$\therefore$ Therefore, the required line is

$$x + \sqrt{3}\, y = 0 + \sqrt{3} \cdot \sqrt{3} = 3$$

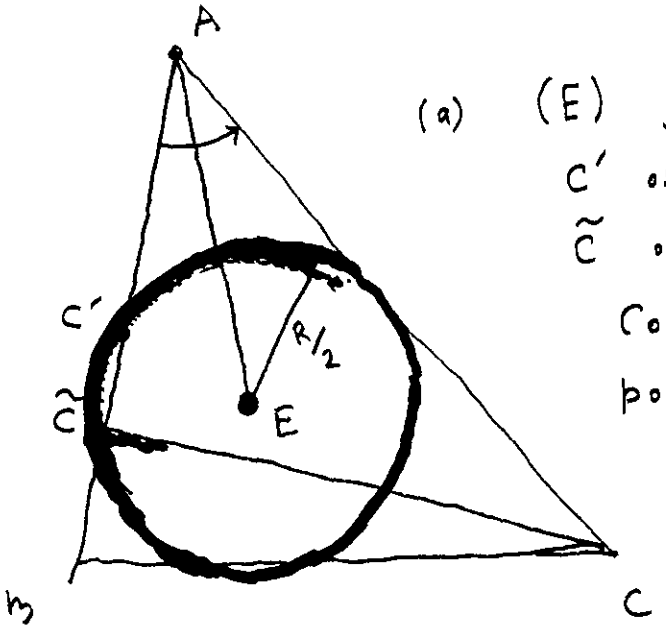
$$\boxed{x + \sqrt{3}\, y = 3}$$

Let $E, (E), R$ and Δ denote the center of the nine point circle, the nine point circle, the circumradius and the area of the triangle ABC respectively.

(a) Compute the power of A with respect to (E).

(b) Show that

$$|AE| = \sqrt{\Delta \cot A + \frac{R^2}{4}} \quad .$$



(a) (E) goes through the midpoint C' of $[A, B]$ and the foot $\tilde{C}$ of the altitude through C.

Consequently the required power is

$$AC' . A\tilde{C} = \frac{b}{2} \cdot c \cdot \cos A$$
$$= \frac{\Delta}{\sin A} \cos A$$
$$= \boxed{\Delta \cot A}$$

(b) Obvious from

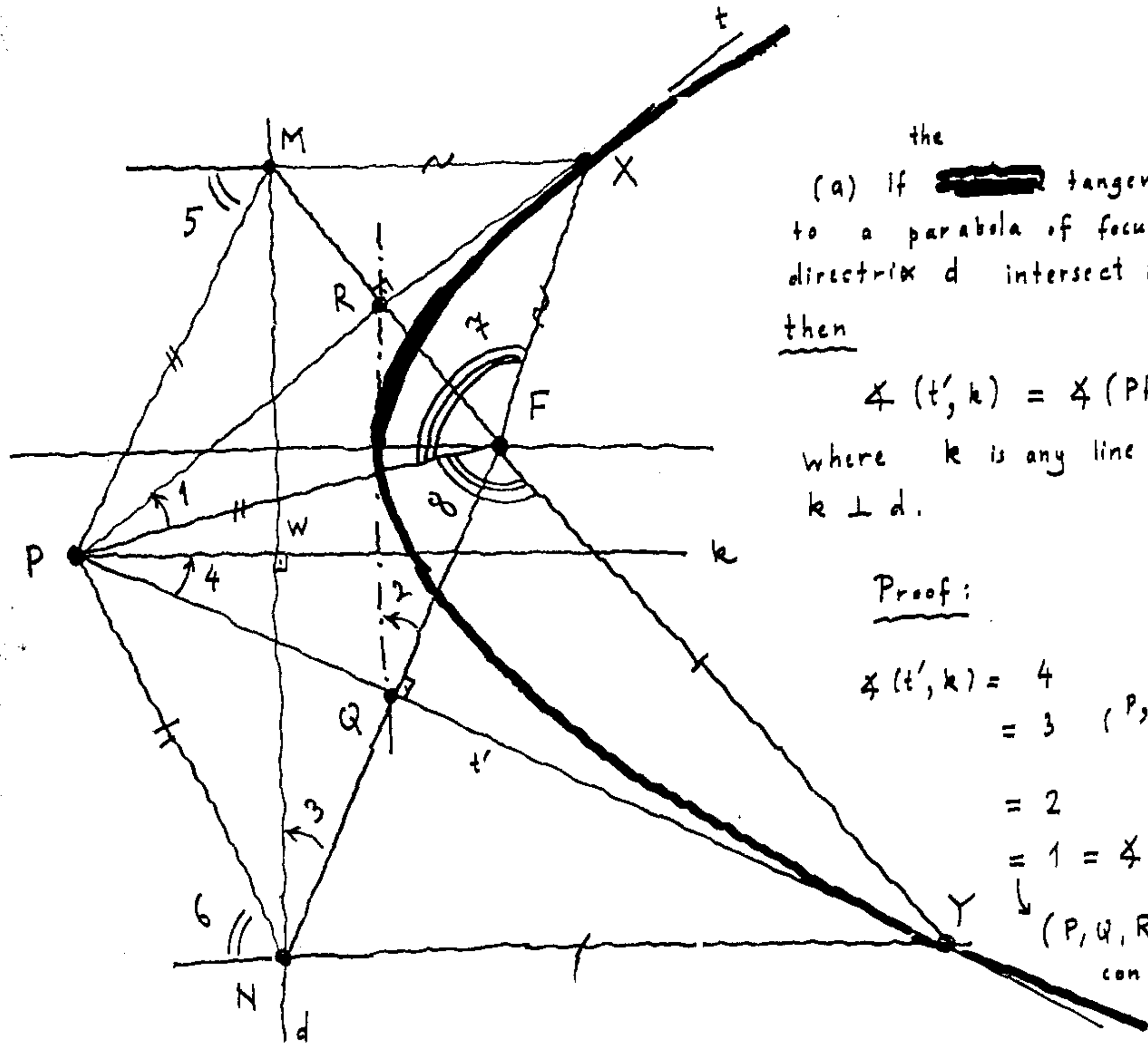
$$|AE|^2 - R^2/4 = \text{the power of } A \text{ with respect to } (E)$$

In solving the following problems the student is <u>not allowed</u> to use the word "infinity" or the sign ∞.

(a) State and prove the first theorem of Poncelet for a parabola.

(b) State and prove the second theorem of Poncelet for a parabola.

(c) On a parabola of focus F, consider points X, Y such that $F \in XY$. Let the tangents to the parabola at X, Y intersect in P. Prove that $XP \perp YP$ and $FP \perp XY$.



(a) If the [redacted] tangents t, t' to a parabola of focus F, directrix d intersect in P

then

$$\measuredangle (t', k) = \measuredangle (PF, t)$$

where k is any line with $k \perp d$.

Proof:

$\measuredangle (t', k) = 4$

$= 3$ (P, W, Q, N concyclic!)

$= 2$

$= 1 = \measuredangle (PF, t)$

(P, Q, R, F concyclic!)

(b) If the tangents to a parabola φ of focus F at $X \in \varphi$, $Y \in \varphi$ intersect in P, then

$$\measuredangle(FX, FP) = \measuredangle(FP, FY)$$

Proof: $\measuredangle(FX, FP) = 7 = \pi - 5 = \pi - 6$
$= 8 = \measuredangle(FP, FY)$.

(c)

By Poncelet 2
$\alpha = \beta$. As $\alpha + \beta = \pi$
we conclude $\alpha = \beta = \pi/2$
Hence $PF \perp XY$.

On the other hand

$3 = 5 = 2 \underset{\text{Poncelet I}}{=} 1$

$= \frac{\pi}{2} - 4$

[redacted]. which shows that PXY is a right triangle...