FINAL EXAMINATION

FAMILY NAME	OTHER NAMES	GRADE

14th January 2013. Duration : 2.5 hours.

Five questions : 5 + 7 + 8 , 18 , 7 + 13 , 17 , 5 + 5 + 5 + 5 + 5

Solutions

1. Consider a triangle ABC with $a = |BC|$, $b = |CA|$, $c = |AB|$ and a point $X \in [B, C]$ with $|AX| = x$, $|BX| = p$, $|CX| = q$. Let m_a, m_b, m_c denote the lengths of the medians through A, B, C.

(A) By employing the cosine rule or otherwise prove the Stewart relation

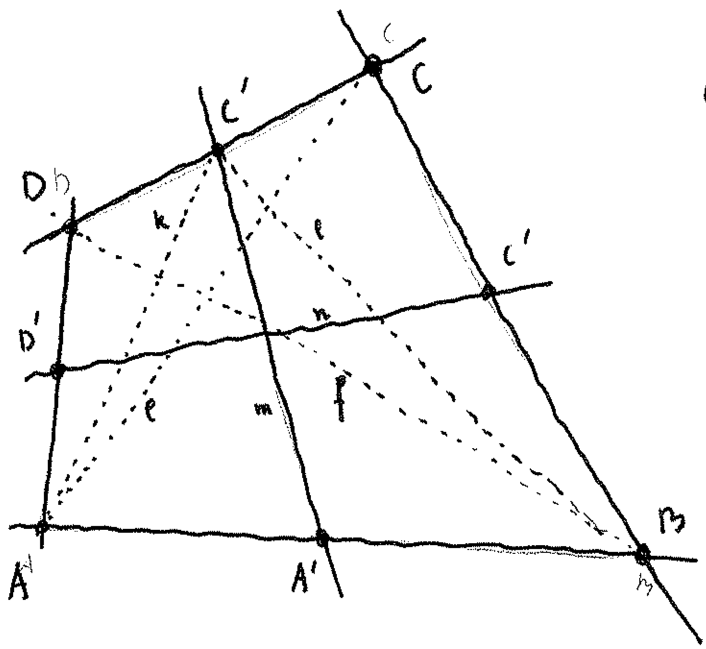
$$ax^2 = pb^2 + qc^2 - apq\ .$$

(B) Show that

$$m_a^2 = \frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}\ .$$

(C) Consider a quadrangle $ABCD$ with diagonals $e = |AC|$ and $f = |BD|$. Let A', B', C', D' be the respective midpoints of $[A, B]$, $[B, C]$, $[C, D]$, $[D, A]$ and let $m = |A'C'|$, $n = |B'D'|$. Prove that

$$m^2 + n^2 = \frac{e^2 + f^2}{2}$$



A, B routine...

(C) Let $k = |AC'|$, $l = |BC'|$

Let $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DA| = d$.

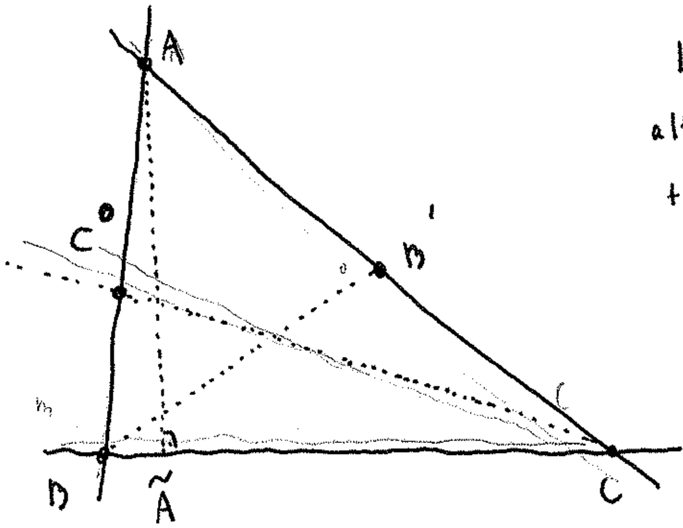
$$m^2 = |A'C'|^2 = \frac{k^2}{2} + \frac{l^2}{2} - \frac{a^2}{4} = \frac{\frac{d^2}{2} + \frac{e^2}{2} - \frac{c^2}{4}}{2} + \frac{\frac{f^2}{2} + \frac{b^2}{2} - \frac{c^2}{4}}{2} - \frac{a^2}{4}$$

$$= \frac{e^2 + f^2 + b^2 + d^2 - a^2 - c^2}{4}$$

Similarly

$$n^2 = \frac{e^2 + f^2 + a^2 + c^2 - b^2 - d^2}{4}\ .$$ Hence...

2. Prove that in a triangle ABC the altitude through A, the median through B and the internal angle bisector through C are concurrent iff $\sin A = \cos B \tan C$.



Let $\tilde{A}$ be the foot of the altitude through A, let B', C^0 be the points in which the median through B and the internal bisector at C intersect CA and AB respectively.

$A\tilde{A}$, BB', CC^0 are concurrent (parallelity is impossible as BB', CC^0 always intersect !) iff

$$\frac{\tilde{A}B}{\tilde{A}C} \cdot \frac{B'C}{B'A} \cdot \frac{C^0A}{C^0B} = \left(-\frac{c\cos(B)}{b\cos(C)}\right)\cdot(-1)\left(-\frac{\overset{\overset{|CA|}{\shortparallel}}{b}}{\underset{\underset{|BC|}{\shortparallel}}{a}}\right) = -1$$

which happens iff

$$\underset{\underset{2R\sin A}{\shortparallel}}{a}\cos C = \overset{\overset{2R\sin C}{\shortparallel}}{c}\cos B$$

iff

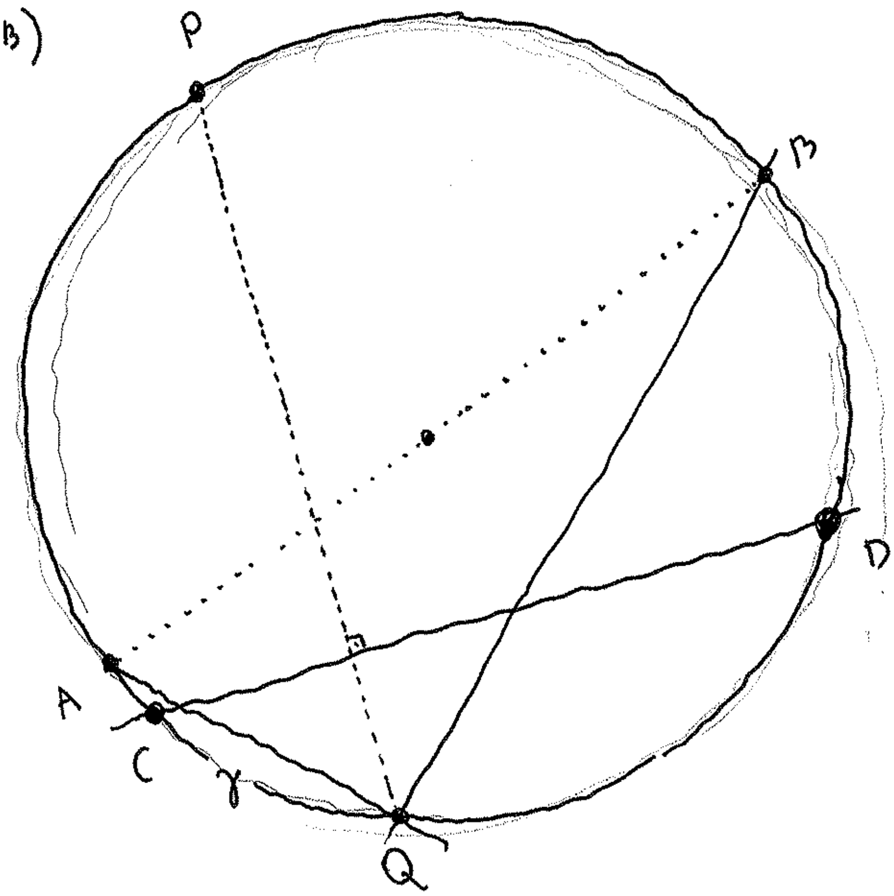
$$\sin A = \cos B \cdot \tan C$$

3. (A) Consider a triangle UVW and a point X on the circumcircle of UVW. Prove that the Simson line of X with respect to UVW is parallel to UY where $[X, Y]$ is the unique chord of the circumcircle of UVW perpendicular to VW.

(B) Given a circle γ, consider points $P, C, D \in \gamma$ and let $[A, B]$ be a diameter of γ. Prove that the Simson line of P with respect to the triangle ACD is perpendicular to the Simson line of P with respect to the triangle BCD.

(A) Standard

(B)



The Simson line of P w.r.to ACD is parallel to AQ

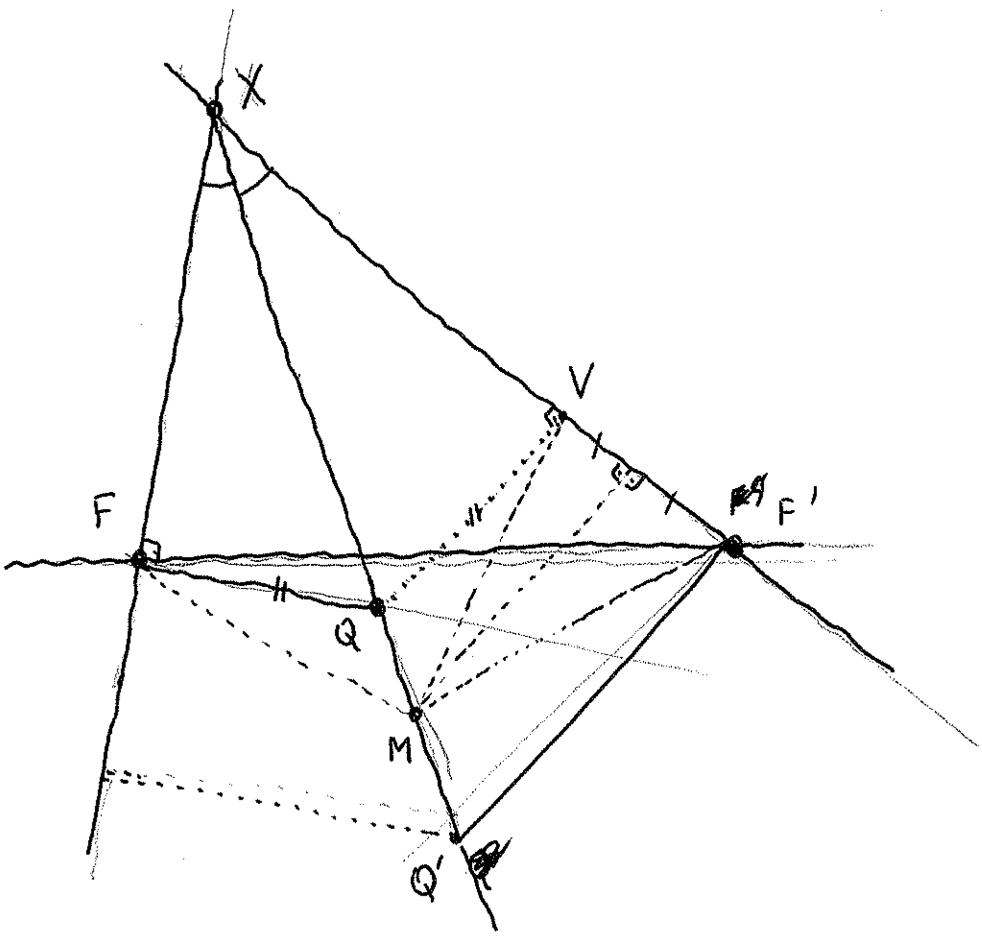
&

the Simson line of P w.r.to BCD is parallel to BQ.

&

Note that AQ ⊥ BQ !

4. Let φ be an ellipse of foci F, F'. Consider $X \in \varphi$. Let Q, Q' be the points in which the normal of φ at X intersects the perpendiculars to XF, XF' erected at F, F' respectively. Prove that, the perpendicular bisector of $[FF']$ bisects $[P, P']$.



Let M be the midpoint of $[Q, Q']$. Argue that

$$|MF| = |MV| = |MF'|$$

why? why?

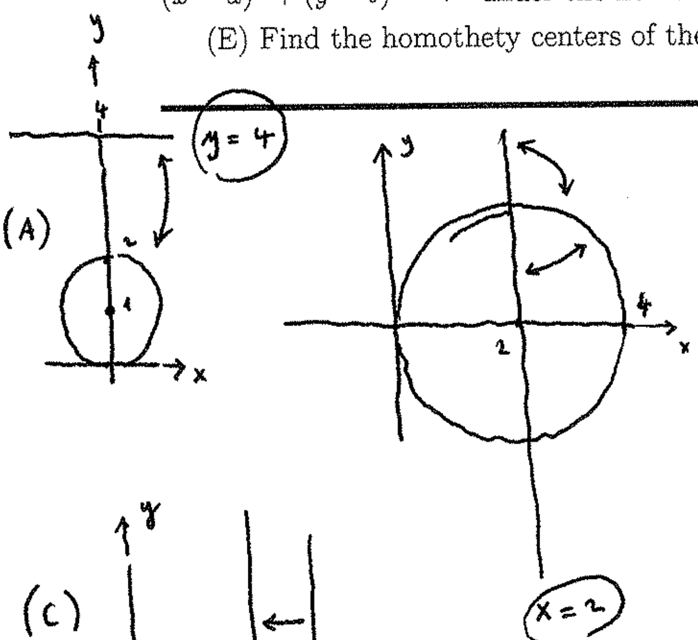
5. (A) What is the image of the circles $x^2 + (y-1)^2 = 1$ and $(x-2)^2 + y^2 = 4$ under the inversion $\mathrm{Inv}\Big((0,0), 8\Big)$?

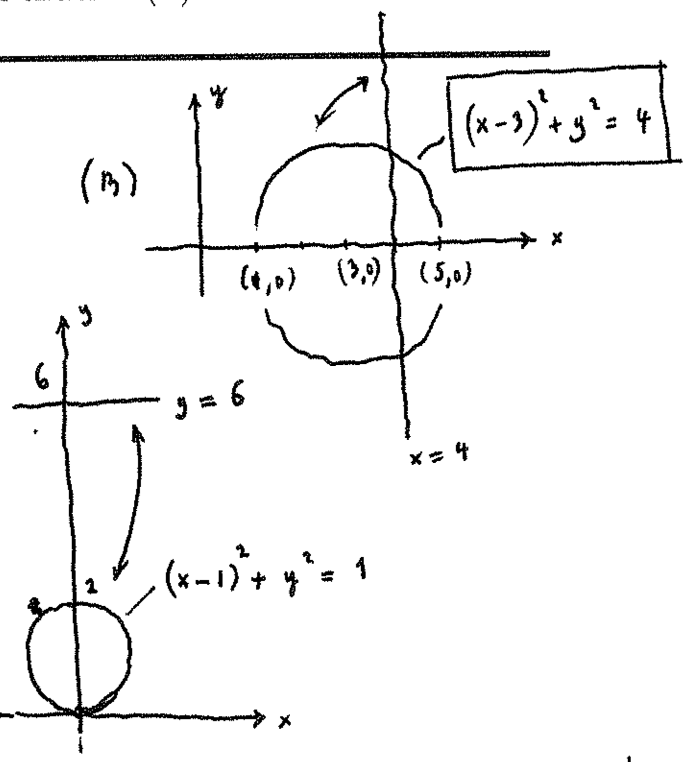
(B) What is the image of the lines $x = 4$ and $y = 6$ under the inversion $\mathrm{Inv}\Big((1,0), 12\Big)$?

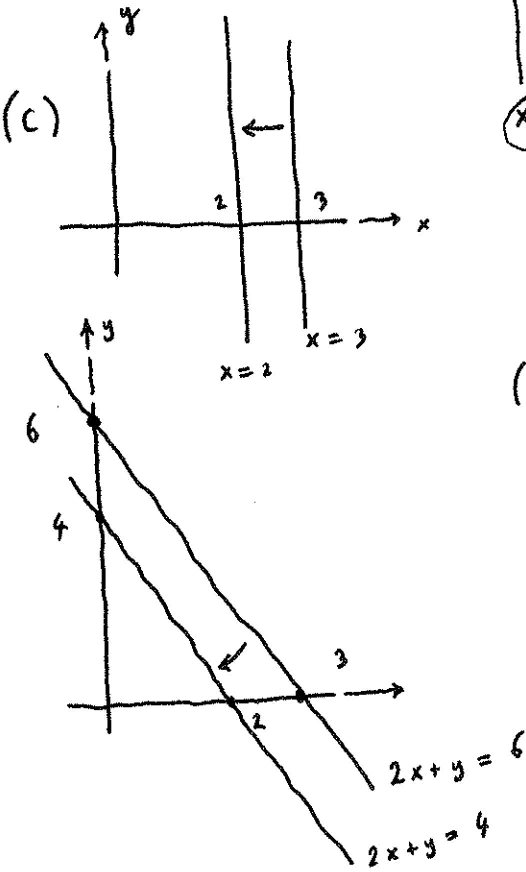
(C) What is the image of the lines $x = 3$ and $2x + y = 6$ under the homothety $\mathrm{Hom}\Big((0,0), 2/3\Big)$?

(D) Write down the equation of the circle which is the image of the circle $(x-a)^2 + (y-b)^2 = r^2$ under the homothety $\mathrm{Hom}\Big((0,0), \lambda\Big)$ where $\lambda \neq 0$.

(E) Find the homothety centers of the circles in (D).





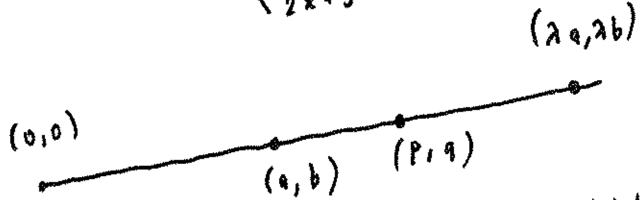


(D) Its center is $(\lambda a, \lambda b)$, its radius is $|\lambda| r$

Consequently it is the circle

$$(x - \lambda a)^2 + (x - y a)^2 = \lambda^2 r^2$$

(E) One of the homothety centers is already known: It is $(0,0)$. The second is the point that divides $[(a,b), (\lambda a, \lambda b)]$ in the ratio $-\lambda$:



$$\frac{p - \lambda a}{p - a} = -\lambda \longrightarrow p - \lambda a = -\lambda p + \lambda a$$

$$\longrightarrow p = \frac{2\lambda}{1+\lambda} a$$

Similarly : $q = \dfrac{2\lambda}{1+\lambda} b$

provided $\lambda \neq -1$. (In this case there is no second center!)