MATH 373 - GEOMETRY I

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

7th November 2012. Duration : 100 minutes. Three questions : $(5+5)+15+10$, $5+10+10+10$, $20+10$

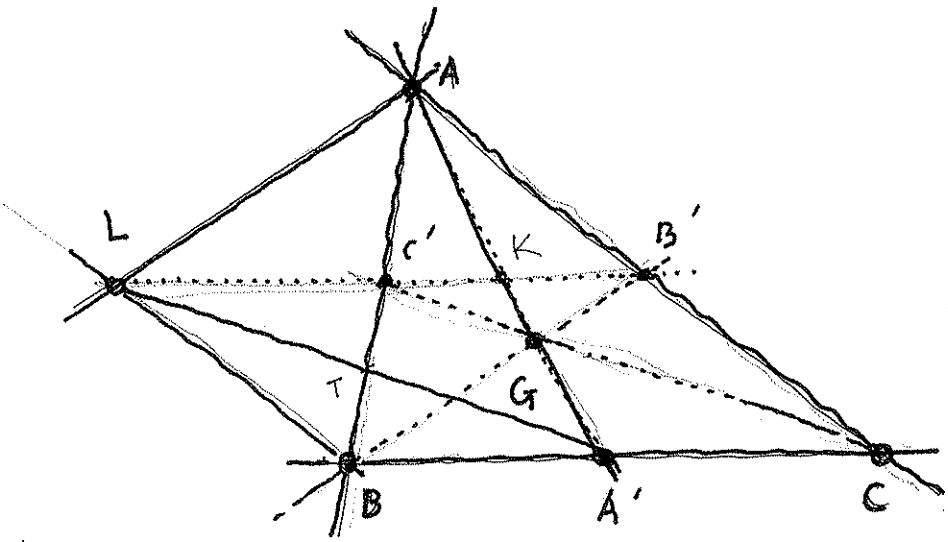
Solutions

1. In a triangle ABC with centroid G, let AG, BG, CG meet BC, CA, AB in A', B', C' respectively. Let the parallel to AC through B intersect $B'C'$ in L. Prove that

(A) AL is parallel to BB' and $|AL| = |BB'|$, $|A'L| = |CC'|$.

(B) Prove that the line BA is the median of the triangle $AA'L$ through the point A.

(C) Prove that the length of the median of the triangle $AA'L$ through the point A is $3c/4$ where $c = |AB|$.



(A) $B'C' \parallel BC$ hence

$BC\,B'L'$ is a parallelogram.

Therefore $|BL| = |B'C| = |B'A|$. As $LB \parallel AC = B'A$ we conclude that $LBB'A$ is a parallelogram. $\therefore$ $AL \parallel BB'$ & $|AL| = |BB'|$.

Similarly note that $|BA'| = |C'B|$ & $|C'B'| = |C'L|$

($LBB'A$ is a parallelogram, diagonals meet in C' !)

Hence $|C'L| = |C'B'| = |A'C|$ &

$LA'CC'$ is a parallelogram.

(B) LC' is the median of LBA hence that of LAA'.

$C'K : C'L = -1 : 2$. Hence C' is the centroid of ALA'.

Therefor $AC' = AB$ is a median of ALA'.

(C) Note that T is the midpoint of $[A, B]$.

2. Given an acute angled positively oriented triangle ABC, let O and H be the circumcenter and the orthocenter of ABC, respectively. Let AH intersect BC in $\tilde{A}$. Let A' be the midpoint of $[B, C]$. Prove that

(A) $|AH| = 2|OA'|$,
(B) $|AH| = 2R\cos A$,
(C) $|\tilde{A}H| = 2R\cos B\cos C$,
(D) OH is perpendicular to AH iff $\tan B\tan C = 3$.

(A) Routine

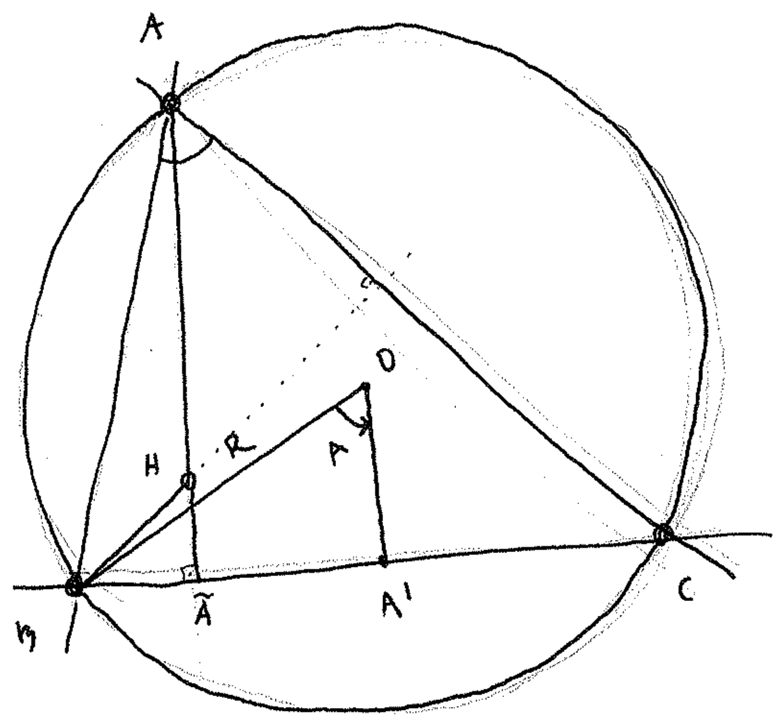
(B) Note : $|OA'| = R\cos A$ acute ...

(C) Similarly $|BH| = 2R\cos B$.

In the right triangle $HB\tilde{A}$

$$|\tilde{A}H| = |BH|\sin\left(\frac{\pi}{2} - C\right)$$

$$= 2R\cos B\cos C$$



(D) $OH \perp AH$ $\underline{\underline{\text{iff}}}$

$$|OA'| = |H\tilde{A}|$$

$$\cancel{R}\cos A = 2\cancel{R}\cos B\cos C$$

$$-\cos(B + C)$$

$$-\cos B\cos C + \sin B\sin C = 2\cos B\cos C \quad ✓.$$

3. Given a pentagon $ABCDE$, consider $X \in AB - \{A, B\}$, $Y \in BC - \{B, C\}$, $Z \in CD - \{C, D\}$, $T \in DE - \{D, E\}$, $U \in EA - \{E, A\}$.

(A) If X, Y, Z, T, U are collinear, Prove that

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TE} \cdot \frac{UE}{UA} = 1 \quad .$$

(B) Is the converse true ?

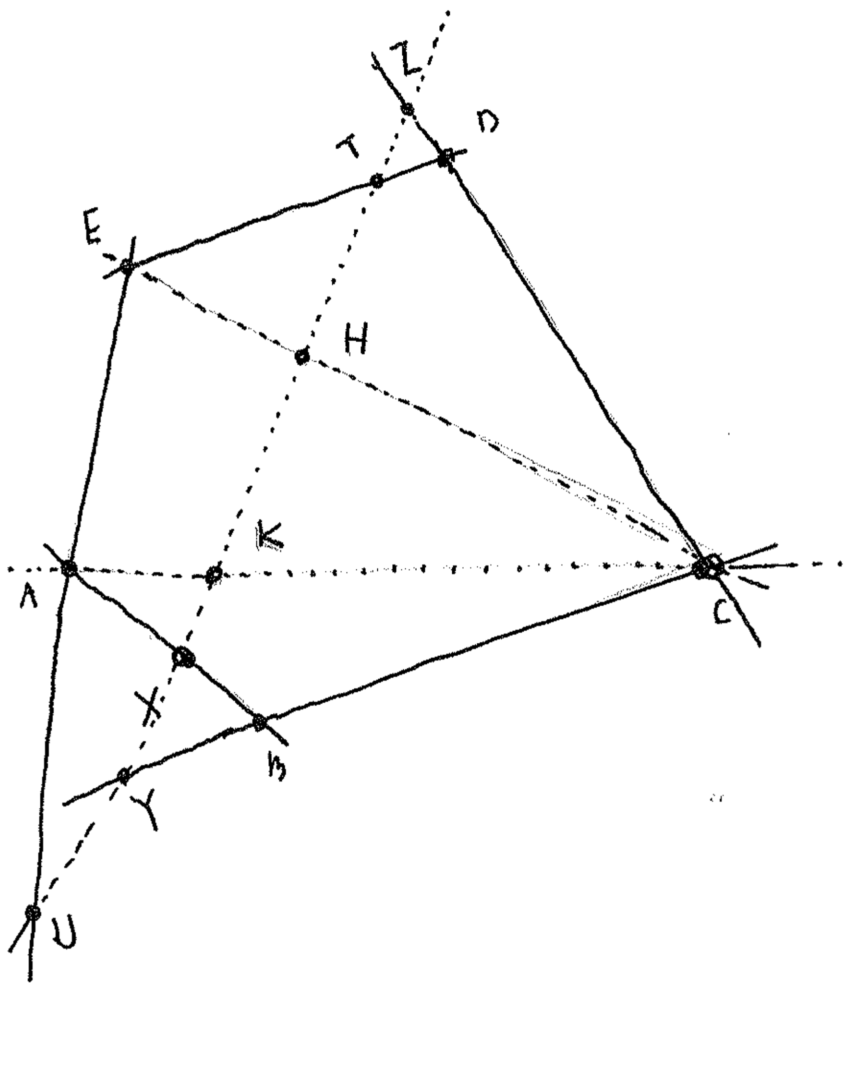
(A) Let X, Y, Z, T, U be collinear. "Menelaus" in

BCA : $\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{KC}{KA} = +1$

ACE : $\frac{KA}{KC} \cdot \frac{HC}{HE} \cdot \frac{UE}{UA} = +1$

ECD : $\frac{HE}{HC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TE} = +1$

Multiply all...



(B) No : Consider midpoints X, Y, Z of $[A, B]$, $[B, C]$, $[C, A]$ respectively.

Choose any $T \in [D, E] - \{D, E\}$ $U \in [E, A] - \{E, A\}$ with

$\frac{TD}{TE} = - \frac{UA}{UE}$. Let $ABCDE$ be a regular pentagon, for instance...