MATH 373 - GEOMETRY I

SECOND MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

19th December 2012. Duration : 100 minutes. Three questions : $10 + 10 + 15$, $15 + 15$, $10 + 5 + 10 + 10$

Solutions

1. (A) Compute the power of the point $P(x_0, y_0)$ with respect to the circle $x^2+y^2+2ax+2by+c=0$.

(B) Write down the equation of the radical axis of the circles $x^2+y^2+2a_1x+2b_1y+c_1=0$ and $x^2+y^2+2a_2x+2b_2y+c_2=0$.

(C) Prove that the set of points whereof the ratio of powers with respect to two given circles Γ_1, Γ_2 is a constant is a circle the centre of which lies on the line joining centres of Γ_1, Γ_2.

(A) The circle $x^2+y^2+2ax+2by+c=0$ is the circle of center $(-a,-b)$ and radius $R=\sqrt{a^2+b^2-c}$. (Therefore $a^2+b^2-c>0$!)

$\therefore$ The power of (x_0, y_0) with respect to is

$$\left[\operatorname{dist}\left((x_0,y_0),(-a,-b)\right)\right]^2-R^2$$

$$=(x_0+a)^2+(y_0+b)^2-(a^2+b^2-c) = \underline{x_0^2+y_0^2+2ax_0+2by_0+c}$$

(B) The radical axis consists of points (x,y) which have the same power with respect to the above circles. Therefore

$$x^2+y^2+2a_1x+2b_1y+c_1 = x^2+y^2+2a_2x+2b_2y+c_2$$

equivalently $\boxed{(a_1-a_2)x+(b_1-b_2)y+c_1-c_2=0}$

which represents a line if $(a_1,b_1)\neq(a_2,b_2)$, that is, if the circles in question are not concentric...

(C) The ~~set of~~ points (x,y) whereof the ratio of powers with respect to Γ_1 (which we take to be $x^2+y^2=1$ for simplicity !) and w.r.to Γ_2 (which we take to be $(x-a)^2+y^2=R^2$ for the same reason...) satisfy

$$\frac{x^2+y^2-1}{(x-a)^2+y^2-R^2}=\lambda$$

$$x^2+y^2-1=\lambda\left[x^2+y^2-2ax+a^2-R^2\right]$$

$$\Leftrightarrow (1-\lambda)(x^2+y^2)+2\lambda ax-1-\lambda a^2+\lambda R^2=0$$

If $\lambda\neq1$.

$$\Leftrightarrow x^2+y^2+\frac{2\lambda a}{1-\lambda}x+\frac{-1-\lambda a^2+\lambda R^2}{1-\lambda}=0$$

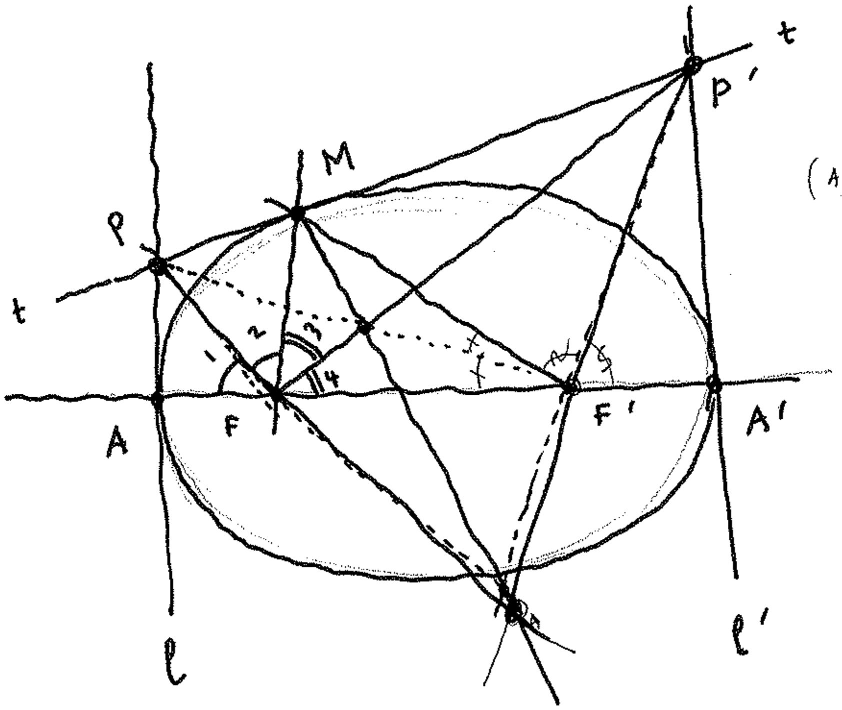
A circle with its center on the x-axis...

If $\lambda=1$, the points constitute a line. A "generalized circle"...

2. Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ. at A, A'. For any tangent t to φ at $M \in \varphi$ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$.

(A)By applying the second theorem of Poncelet first to the tangent lines t, l and then to the tangent lines t, l', or otherwise, prove that $<(FP, FP') =< (F'P, F'P') = \pi/2$.

(B) Prove that $FP, F'P'$ intersect on the normal to φ at M.



(A) Briefly "1" = "2" & "3" = "4".

It follows that

$\measuredangle(FP, FP') = "2" + "3" = \pi/2$

Similarly.

$\measuredangle(F'P, F'P') = \pi/2$

(B) In the triangle MFF', the lines PF and $P'F'$ are external bisectors. Hence they intersect on the internal bisector of $\measuredangle FMF'$ which is the normal to φ at M – why?

3. Consider a convex positively oriented quadrangle $ABCD$ with positively oriented squares $BAPQ$, $CBRS$, $DCTU$, $ADVW$ on its sides with respective centers M_a, M_b, M_c, M_d. Let J be the midpoint of $[A, C]$.

(A) Prove that

$$\Phi = \mathrm{Rot}(M_a, \frac{\pi}{2}) \circ \mathrm{Rot}(M_b, \frac{\pi}{2})$$

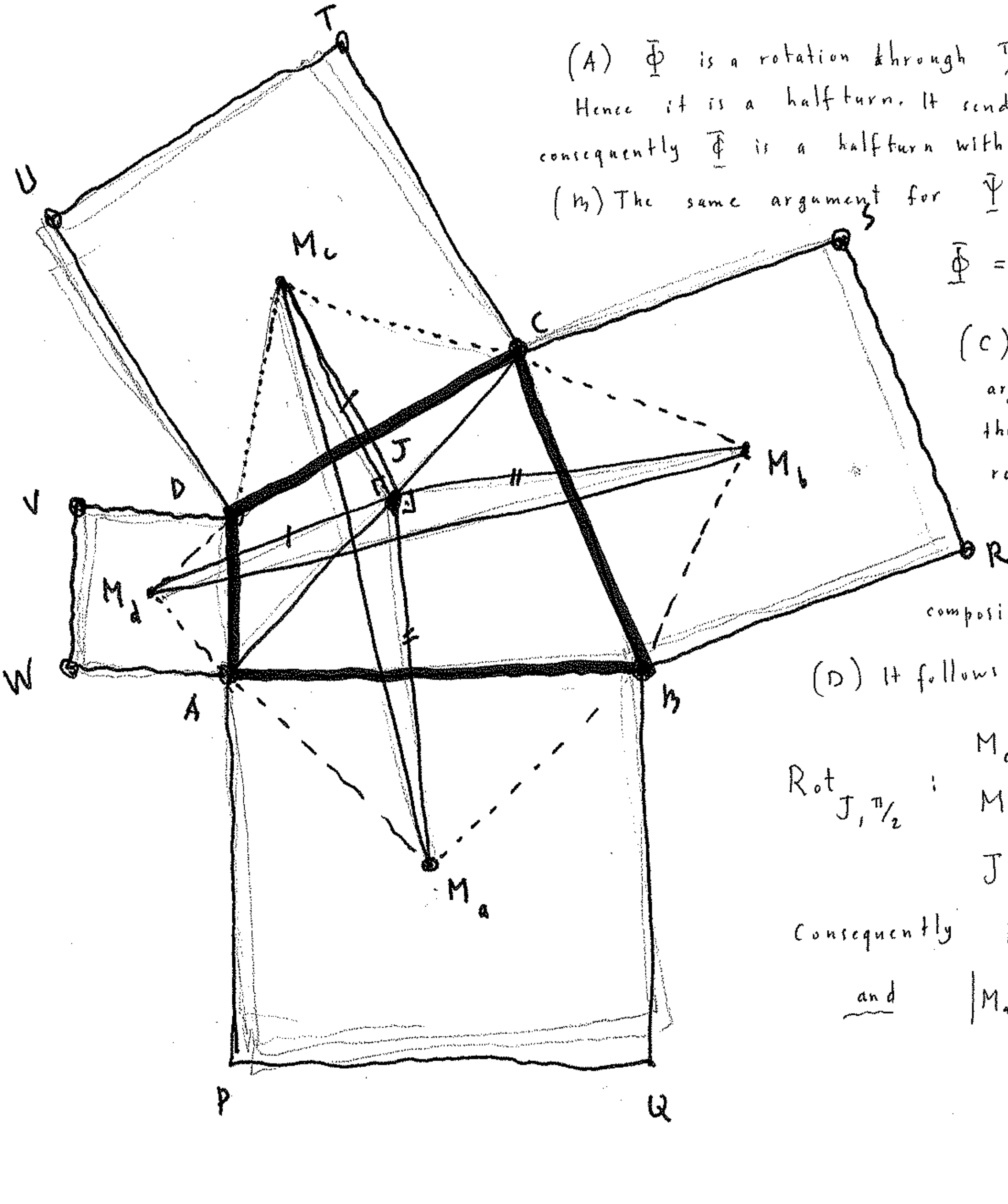
is a halfturn with center J.

(B) What can you say about the isometry

$$\Psi = \mathrm{Rot}(M_c, \frac{\pi}{2}) \circ \mathrm{Rot}(M_d, \frac{\pi}{2}) \quad ?$$

(C) Prove that the triangles JM_aM_b and JM_cM_d are right isosceles triangles.

(D) Prove that $M_aM_c \perp M_bM_d$ and $|M_aM_c| = |M_bM_d|$.



(A) Φ is a rotation through $\pi/2 + \pi/2 = \pi$. Hence it is a halfturn. It sends C into A consequently Φ is a halfturn with center J.

(B) The same argument for Ψ. That is

$$\Phi = \Psi = HT_J .$$

(C) The standard argument for finding the center of a rotation (in this case a half-turn!) which is the composition of two rotations...

(D) It follows that

$$\mathrm{Rot}_{J, \pi/2} : \quad \begin{array}{l} M_c \longrightarrow M_d \\ M_a \longrightarrow M_b \\ J \longrightarrow J \end{array}$$

Consequently $M_aM_c \perp M_bM_d$

and $|M_aM_c| = |M_bM_d|$.