MATH 373 - GEOMETRY I

FINAL EXAMINATION

FAMILY NAME	OTHER NAMES	GRADE

20th January 2014. Duration : 150 minutes.
Five questions : 5 + 12 + 3 , 12 + 8 , 7 + 5 + 8 , 14 + 6 , 4 + 4 + 4 + 4 + 4

Solutions

1. (A) Given a triangle ABC and $T \in [B,C]$, prove the Stewart Relation

$$ax^2 = pb^2 + qc^2 - apq$$

where $a = |BC|$, $b = |CA|$, $c = |AB|$, $x = |AT|$, $p = |BT|$, $q = |TC|$, by employing the cosine rule or otherwise.

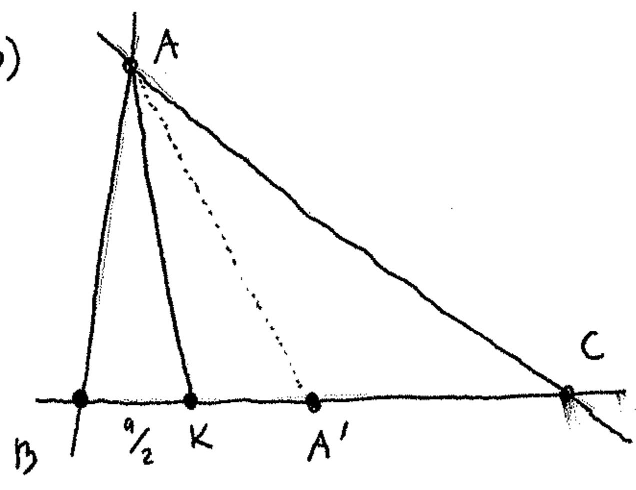
(B) Let A' be the midpoint of $[B,C]$ and let K, L be the respective midpoints of $[B,A']$, $[A',C]$. Prove that

$$|AK| = \sqrt{\frac{b^2}{4} + \frac{3c^2}{4} - \frac{3a^2}{16}}\,.$$

(C) Compute $|AL|$.

(A) ✓

(B)



Put $p = a/4 \quad q = \frac{3a}{4}$

to obtain

$$a|AK|^2 = \frac{a}{4}b^2 + \frac{3a}{4}c^2 - a\frac{a}{4}\frac{3a}{4}$$

$\longrightarrow$ desired result.

2. Given a triangle ABC, let A', B', C' be the respective midpoints of $[B,C]$, $[C,A]$, $[A,B]$. Let $K \in BC$, $L \in CA$, $M \in AB$ be the respective reflections of B', C', A' in the internal bisectors of C, A, B. Let $a = |BC|$, $b = |CA|$, $c = |AB|$.

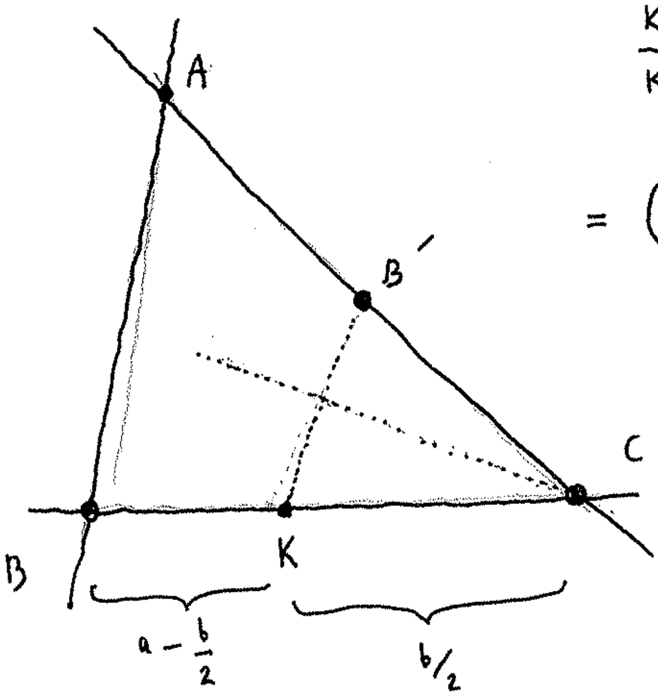
(A) Prove that AK, BL, CM are concurrent or parallel iff

$$\frac{2b-c}{a} + \frac{2c-a}{b} + \frac{2a-b}{c} = 3\,.$$

(B) Prove that $\cancel{A}K$, $\cancel{B}L$, $\cancel{C}M$ are collinear iff

$$\frac{2b-c}{a} + \frac{2c-a}{b} + \frac{2a-b}{c} = 4\,.$$

(A) AK, BL, CM are concurrent or parallel iff



$$\frac{KB}{KC} \cdot \frac{LC}{LA} \cdot \frac{MA}{MB}$$

$$= \left(-\frac{a - b/2}{b/2}\right)\left(-\frac{b - c/2}{c/2}\right)\left(-\frac{c - a/2}{a/2}\right)$$

$= -1$. Equivalently

$$(2a-b)(2b-c)(2c-a) = abc$$

$\rightarrow$ Regroup.

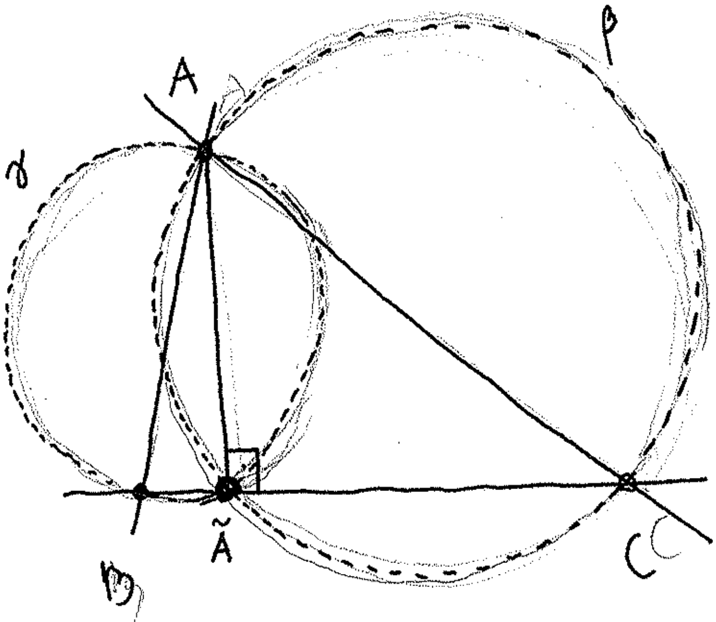
(B) The same with -1 on RHS replaced with $+1$!

3. Given a triangle ABC, let α, β, γ be the circles of respective diameters $[B, C]$, $[C, A]$, $[A, B]$.

(A) Prove that the radical axis of β and γ is the altitude through A.

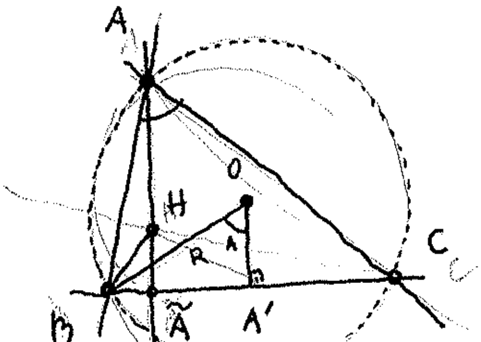
(B) Prove that the radical center of α, β, γ is the orthocenter H of ABC.

(C) Prove that the power of H is $-4R^2 \cos A \cos B \cos C$ with respect to any of α, β, γ.



(A) If $\tilde{A}$ is the foot of the perpendicular from A onto BC, then β and γ go through $\tilde{A}$.
$\therefore$ $A\tilde{A}$, i.e. the altitude through A is the radical axis of β and γ

(B) Three altitudes concur in H, hence...



Let A' be the midpoint of $[B,C]$

(C) $|HA| = 2|OA'|$

$= 2R\cos A$

Similarly $|HB| = 2R\cos B$ and

$|H\tilde{A}| = |HB|\cos C = 2R\cos B\cos C$

$HA \cdot H\tilde{A} = -4R^2 \cos A \cos B \cos C$.

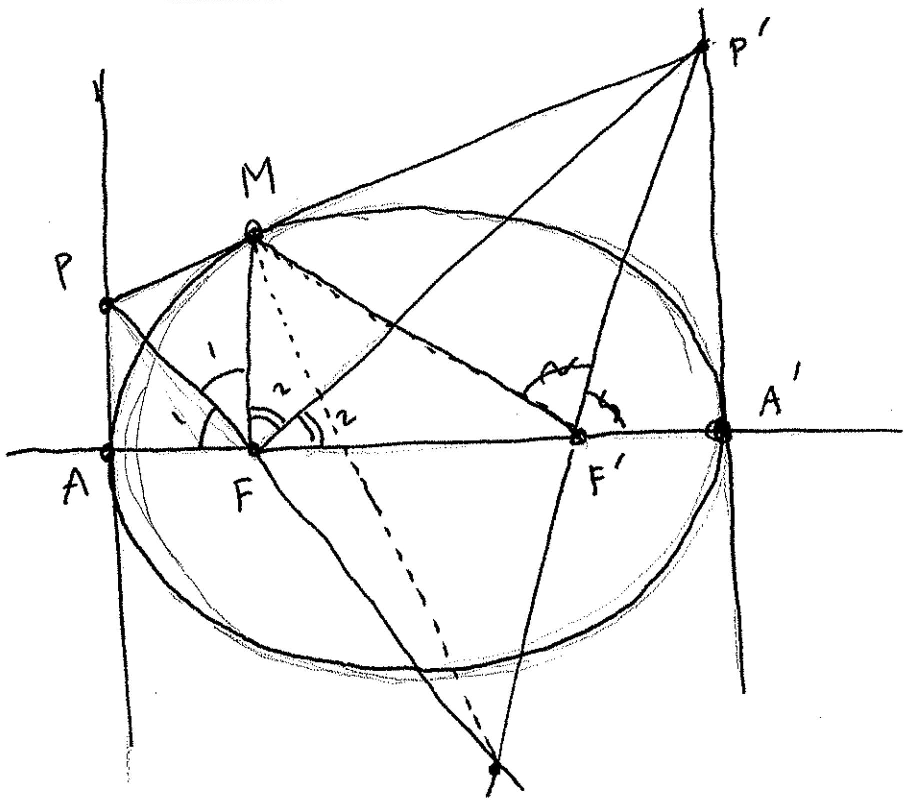
4. Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ. at A, A'. For any tangent t to φ at $M \in \varphi$ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$.

(A) Prove that

$$< (FP, FP') =< (F'P, F'P') = \pi/2$$

(*Hint* : Apply the second theorem of Poncelet to the tangent lines t, l first and then to the tangent lines t, l' .)

(B) Prove that $FP, F'P'$ intersect on the normal to φ at M. (*Hint* : The normal at M is the internal bisector of FMF' at M.)



(A) Note that $2\left(\text{"}1\text{"} + \text{"}2\text{"}\right) = \pi$!

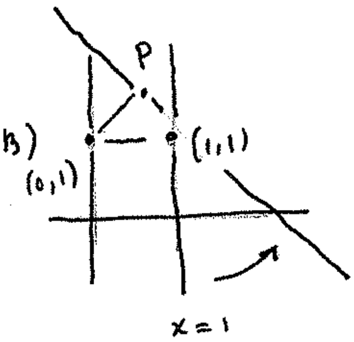
(B) PF, PF' are external bisectors in MFF' hence...

5. (A) What is the image of the line $2x + 3y = 2$ under TR_{u} where $\mathrm{u} = [-2, 5]$?
(B) What is the image of the line $x = 1$ under $\mathrm{Rot}_{(0,1),\pi/4}$?
(C) Given arbitrary points P, Q prove that half-turns in P and Q together give rise to a translation. Which ?
(D) What is the image of the circle $x^2 + y^2 - 2x - 8 = 0$ under $\mathrm{HT}_{(5,1)}$?
(E) What is the image of the line $x + 2y - 5 = 0$ under $\mathrm{Hom}_{(1,1),3}$?

(A) $(1,0)$ is on the line. Hence $(1-2, 0+5) = (-1,5)$ is on the image. $\therefore$ The image is $2(x+1) + 3(y-5) = 0$

$\boxed{2x + 3y = 13}$

(B)



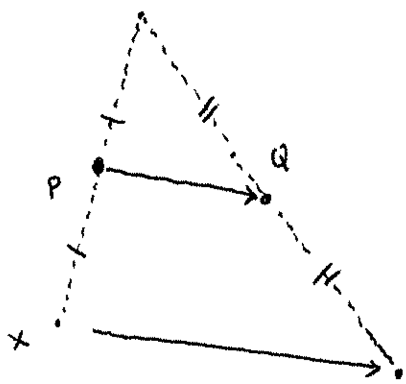
$(1,1)$ is on the line. Its image is $P = \left(\frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right)$

$\therefore$ The image is

$\left(x - \frac{1}{\sqrt{2}}\right) + \left(y - 1 - \frac{1}{\sqrt{2}}\right) = 0$

$\boxed{x + y = 1 + \sqrt{2}}$

(C)

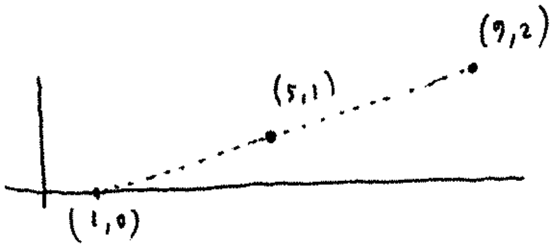


$\mathrm{HT}_Q \circ \mathrm{HT}_P(X) = \mathrm{TR}_{\overrightarrow{PQ}}(X)$

(D) $x^2 + y^2 - 2x - 8 \longrightarrow (x-1)^2 + y^2 = 9$

$\downarrow \mathrm{HT}_{(5,1)}$

$\boxed{(x-9)^2 + (y-2)^2 = 9}$



(E) $(5,0)$ is on the line in question. $(5,0) \xrightarrow{\mathrm{Hom}_{(1,1),3}} (-7,3)$

The image is $(x+7) + 2(y-3) = 0$

$\boxed{x + 2y + 1 = 0}$