MATH 373 - GEOMETRY I

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

30th October 2013. Duration : 100 minutes. Three questions : 10 + 10 + 15 , 15 + 20 , 20 + 10

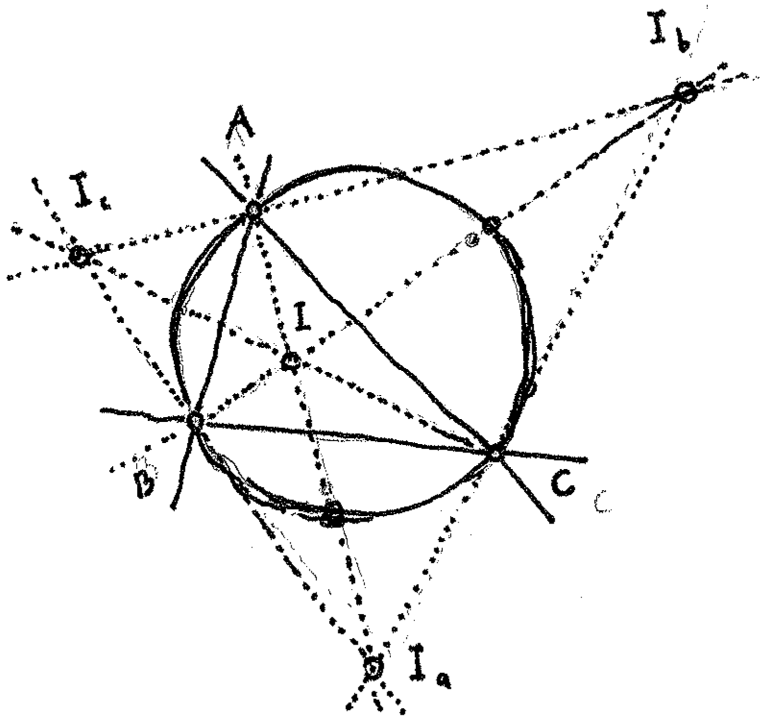
Solutions

1. Let ABC be a triangle with incenter I and excenters I_a; I_b, I_c.

(A) Prove that I is the orthocenter of the triangle $I_aI_bI_c$.

(B) Prove that the circumcircle of a triangle passes through the midpoint of a line segment joining any two of the excenters.

(C) Prove that the circumcircle of a triangle passes through the midpoint of a line segment joining the incenter with any one of the excenters.



(A) $\left.\begin{array}{l} II_a = AI_a \text{ is the internal bisector of } A \\ I_bI_c \quad \text{" " external " " "} \end{array}\right\} \Longrightarrow AI_a \perp I_bI_c$

Similarly $BI_b \perp I_cI_a$ & $CI_c \perp I_aI_b$. Therefore I is the orthocentre of $I_aI_bI_c$. (Better: "The points I, I_a, I_b, I_c constitute and orthic group")

$\left.\begin{array}{l}(B)\\ \&\\ (C)\end{array}\right\}$ Consequently, the circumcircle of ABC is the "9-point-circle" in the "orthic group" I, I_a, I_b, I_c !

2. (A) In a triangle ABC prove that $A = 90°$ iff $r + r_b + r_c = r_a$.

(B) Compute the angle A in a triangle ABC in which

$$3r + r_b + r_c = 3r_a \ .$$

(A) $r_b + r_c = r_a - r \xrightarrow[\text{in view of } \Delta = r \cdot s = r_a(s-a) \text{ etc.}]{} \dfrac{1}{s-b} + \dfrac{1}{s-c} = \dfrac{1}{s-a} - \dfrac{1}{s}$

$$\downarrow$$

$$\frac{a}{(s-b)(s-c)} = \frac{a}{s(s-a)}$$

$$\downarrow$$

$$[a-(b-c)][a+(b-c)] = [(b+c)+a][(b+c)-a]$$

$$a^2 - (b-c)^2 = (b+c)^2 - a^2$$

$$b^2 + c^2 = a^2 .$$

(B) Similarly regrouping

$$\frac{1}{s-b} + \frac{1}{s-c} = \frac{3}{s-a} - \frac{3}{s}$$

one obtains $b^2 + c^2 - a^2 = bc \rightarrow \cos A = 1/2$

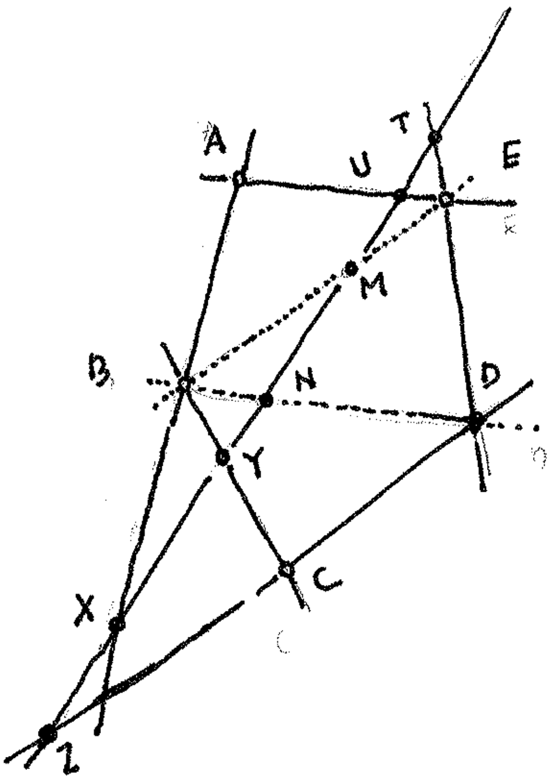
$$\rightarrow A = \pi/3 .$$

3. Given a pentagon $ABCDE$, consider $X \in AB - \{A, B\}$, $Y \in BC - \{B, C\}$, $Z \in CD - \{C, D\}$, $T \in DE - \{D, E\}$, $U \in EA - \{E, A\}$.

(A) If X, Y, Z, T, U are collinear, Prove that

$$\frac{XA}{XB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} \cdot \frac{TD}{TE} \cdot \frac{UE}{UA} = 1 \quad .$$

(B) Is the converse true ?



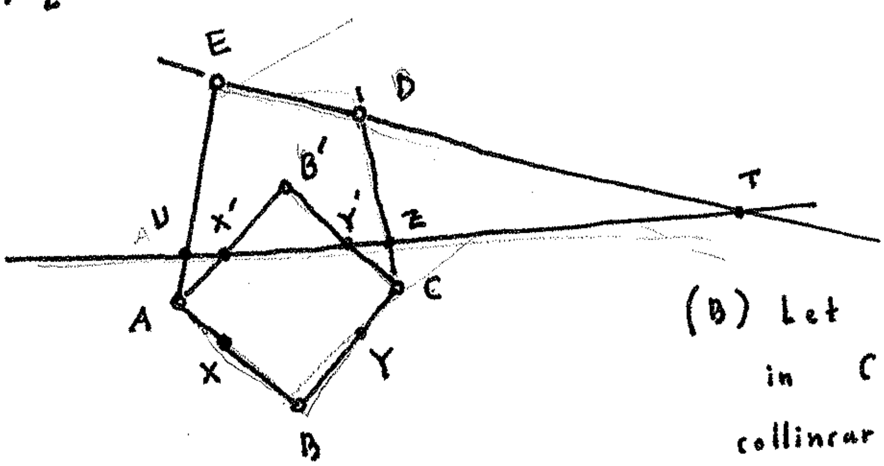
(A) Apply the theorem of Menelaus in the triangles ABE, BDE, BCD to obtain

$$\frac{XA}{XB} \cdot \frac{MB}{ME} \cdot \frac{UE}{UA} = +1$$

$$\frac{ME}{MB} \cdot \frac{NB}{ND} \cdot \frac{TD}{TE} = +1$$

$$\frac{ND}{NB} \cdot \frac{YB}{YC} \cdot \frac{ZC}{ZD} = +1$$

... Multiplying out ...



(B) Let B' be the reflection of B in CA. Let X', Y', Z, T, U be collinear. Let X, Y be the reflection of X', Y' respectively, in CA ...