MATH 373 - GEOMETRY I

SECOND MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

2nd December 2013. Duration : 110 minutes. Three questions : 20 + 15 , 15 + 15 , 10 + 15 + 10

Solutions

1. (A) Compute the power of the point $P(x_0, y_0)$ with respect to the circle $x^2 + y^2 + 2ax + 2by + c = 0$. Write down the equation of the radical axis of the circles $x^2 + y^2 + 2a_1x + 2b_1y + c_1 = 0$ and $x^2 + y^2 + 2a_2x + 2b_2y + c_2 = 0$.

(B) Consider circles Γ and Δ which are tangent to the line k at the points $C, D \in k$ respectively. Prove that the radical axis of Γ and Δ bisects the line segment $[C, D]$.

(A) $\left.\begin{array}{l} x^2+y^2+2ax+2by+c=0 \\ \longrightarrow (x+a)^2+(y+b)^2 = a^2+b^2-c = R^2 \end{array}\right\}$ This is the circle with center $O(-a,-b)$ and radius $R = (a^2+b^2-c)^{1/2}$

$\therefore$ The power of $P(x_0,y_0)$ with respect to this circle

$$= |PO|^2 - R^2 = (x_0+a)^2+(y_0+b)^2 - R^2 = \underline{x_0^2+y_0^2+2ax_0+2by_0+c}$$

The radical axis $= \{(x,y) \mid x^2+y^2+2a_1x+2b_1y+c_1 = x^2+y^2+2a_2x+2b_2y+c_2\}$

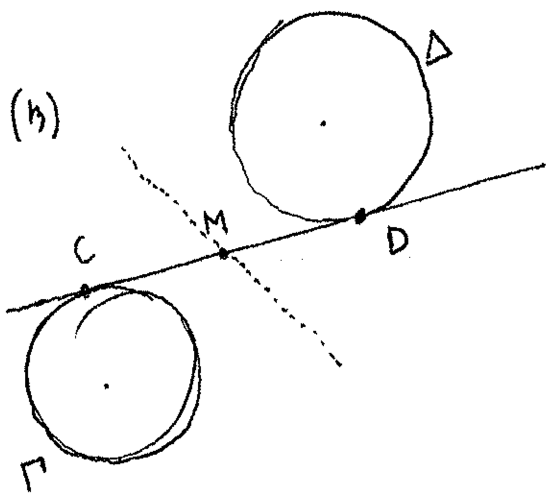
$\therefore$ The equation of the radical axis is

$$(a_1-a_2)x + (b_1-b_2)y + c_1 - c_2 = 0$$

(of course, provided the circles are non-concentric, that is

$$(a_1-a_2, b_1-b_2) \neq (0,0).$$

(B)



If M is the midpoint of $[C,D]$ then

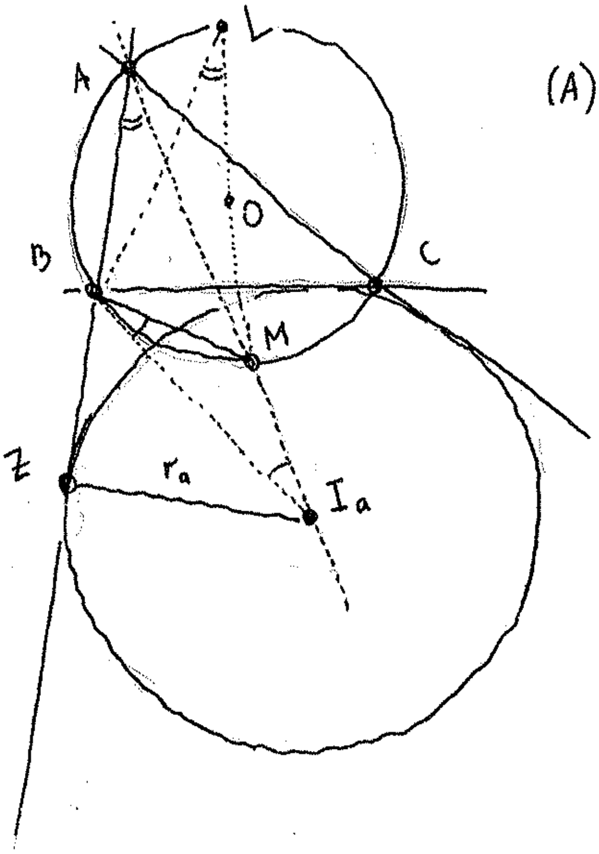
$\left.\begin{array}{c}\text{the power} \\ \text{of } M \\ \text{w.r.to } \Gamma\end{array}\right\} = |MC|^2 = |MD|^2 = \left\{\begin{array}{l}\text{the power} \\ \text{of } M \\ \text{w.r.to } \Delta.\end{array}\right.$

2. (A) Consider a triangle ABC with circumcenter O, incenter I excirles I_a, I_b, I_c, circumradius R, exradii r_a, r_b, r_c. Prove that

$$|OI_a|^2 = R^2 + 2Rr_a \,.$$

(B) Prove that $A = \pi/2$ iff

$$|OI_a|^2 + |OI|^2 = |OI_b|^2 + |OI_c|^2 \,.$$



(A) First notice that MBI_a is an isosceles triangle since

$$\measuredangle I_aBM = \measuredangle MI_aB \,.$$

Consequently, the power of I_a with respect to the circumcircle is

$$|OI_a|^2 - R^2 = I_aM \cdot I_aA \overset{!}{=} |I_aM| \cdot |I_aA| = |I_aZ| \cdot |LM| = r_a \cdot 2R \,.$$

as $LBM \cong AZI_a$

It follows that $|OI_a|^2 = R^2 + 2Rr_a$.

(B) From similar formulae and $|OI|^2 = R^2 - 2Rr$, it is found that

$|OI_a|^2 + |OI|^2 = |OI_b|^2 + |OI_c|^2$ <u>iff</u> $r_a - r = r_b + r_c$

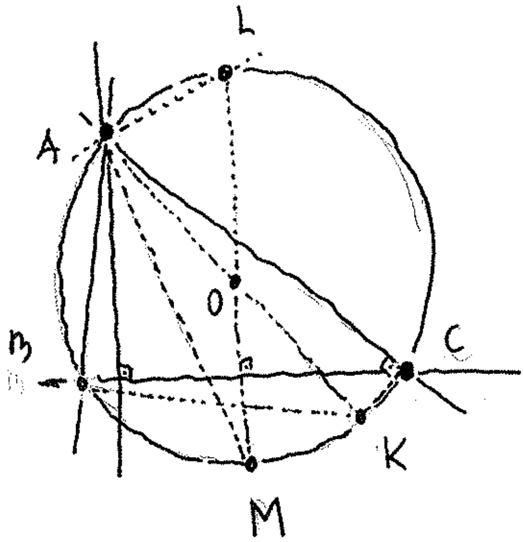
which holds iff $a^2 = b^2 + c^2$!

3. Consider a triangle ABC.

(A) What is the Simson line of A?

(B) Let $[A, K]$ be a diameter of the circumcircle of ABC. Show that BC is the Simson line of K.

(C) Let M be the point on the circumcircle such that AM is the internal bisector of the angle A. Show that the Simson line of M is perpendicular to AM.



(A) The altitude through A.

(B) $KC \perp CA$ & $KB \perp BA$. It follows that the Simson line of K is BC.

(C) The perpendicular from M onto BC is LM, a diameter. The Simson line of L is parallel to AL and $AL \perp AM$.