MATH 373 - GEOMETRY I

FINAL EXAMINATION

FAMILY NAME	OTHER NAMES	GRADE

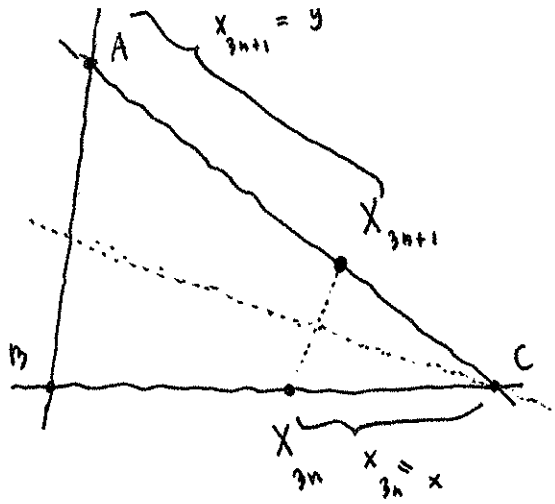
12th January 2015. Duration : 150 minutes.
Five questions : $12 + 8$, $7 + 13$, $12 + 8$, $14 + 6$, $4 + 4 + 4 + 4 + 4$

Solutions

1. Consider a triangle ABC with incenter I.

(A) Given X_n for each $n \in \mathbb{Z}$ with $X_{3n} \in BC$, $X_{3n+1} \in CA$, $X_{3n+2} \in AB$ such that $IC \perp X_{3n}X_{3n+1}$, $IA \perp X_{3n+1}X_{3n+2}$, $IB \perp X_{3n+2}X_{3n+3}$ for all $n \in \mathbb{Z}$, prove that $X_n = X_{n+6}$ for all $n \in \mathbb{Z}$.

(B) Find the unique point $X_0 = S \in BC$ with the property that $X_3 = X_0$ What is the relationship between I and the points $X_0 = S \in BC$ $X_1 = T \in CA$ $X_2 = U \in AB$?



$$\overset{= y}{X_{3n+1}} = b - \overset{= x}{X_{3n}}$$

$$\overset{= z}{X_{3n+2}} = c - y = c - (b - x)$$

$$\overset{= x'}{X_{3n+3}} = a - z = a - c + b - x$$

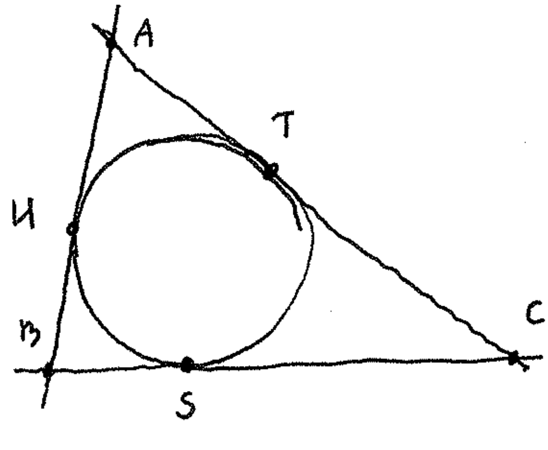
Clearly

$$X_{3n+6} = a - c + b - X_{3n+3} = X_{3n}$$

Hence $X_{3n+6} = X_{3n}$

The same argument with $3n$ replaced with $3n+1$, $3n+2$.

$X_3 = X_0$ iff $x = a - c + b - x$ equivalently $x = s - c$. It follows that $X_0 = S$ is the point in which the incircle touches BC. Similarly T, U are the points in which the incircle touches CA, AB.



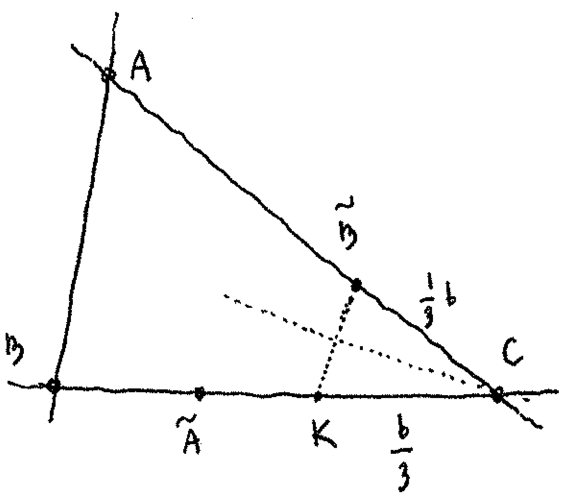
2. Given a triangle ABC, let $\widehat{A}, \widehat{B}, \widehat{C}$ be the points which divide the line segments $[B,C]$ $[C,A]$ $[A,B]$ in the ratio $-1:2$ respectively. Let $K \in BC$, $L \in CA$, $M \in AB$ be the respective reflections of $\widehat{B}, \widehat{C}, \widehat{A}$ in the internal bisectors of C, A, B.

(A) Let $a = |BC|$, $b = |CA|$, $c = |AB|$ and prove that AK, BL, CM are concurrent or parallel iff

$$\frac{3b-c}{a} + \frac{3c-a}{b} + \frac{3a-b}{c} = \frac{25}{3}.$$

(B) Prove that K, L, M are collinear iff

$$\frac{3b-c}{a} + \frac{3c-a}{b} + \frac{3a-b}{c} = 9.$$



(A) AK, BL, CM are concurrent or parallel (parallelity does not come into question!) iff

$$-1 = \frac{KB}{KC}\cdot\frac{LC}{LA}\cdot\frac{MA}{MB} = \left(-\frac{a-\frac{b}{3}}{\frac{b}{3}}\right)\left(\frac{b-\frac{c}{3}}{\frac{c}{3}}\right)\left(-\frac{c-\frac{a}{3}}{\frac{a}{3}}\right)$$

equivalently

$$abc = (3a-b)(3b-c)(3c-a).$$

Regroup to obtain the required identity.

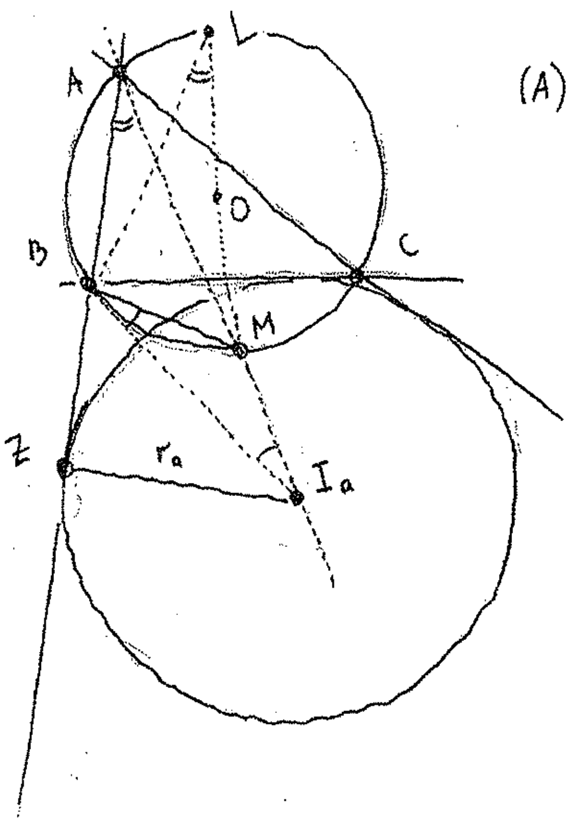
(B) The same reasoning with $-abc = (3a-b)(3b-c)(3c-a)$.

3. (A) Consider a triangle ABC with circumcenter O, incenter I excirles I_a, I_b, I_c, circumradius R, exradii r_a, r_b, r_c. Prove that

$$|OI_a|^2 = R^2 + 2Rr_a\,.$$

(B) Prove that $A = \pi/2$ iff

$$|OI_a|^2 + |OI|^2 = |OI_b|^2 + |OI_c|^2\,.$$



(A) First notice that MBI_a is an isosceles triangle since

$$\measuredangle I_aBM = \measuredangle MI_aB\,.$$

Consequently, the power of I_a with respect to the circumcircle is

$$\begin{aligned} |OI_a|^2 - R^2 &= I_aM \cdot I_aA \\ &\overset{!}{=} |I_aM| \cdot |I_aA| \\ &= |I_aZ| \cdot |LM| \\ &= r_a \cdot 2R. \end{aligned}$$

as $LBM \cong AZI_a$

It follows that $|OI_a|^2 = R^2 + 2Rr_a$.

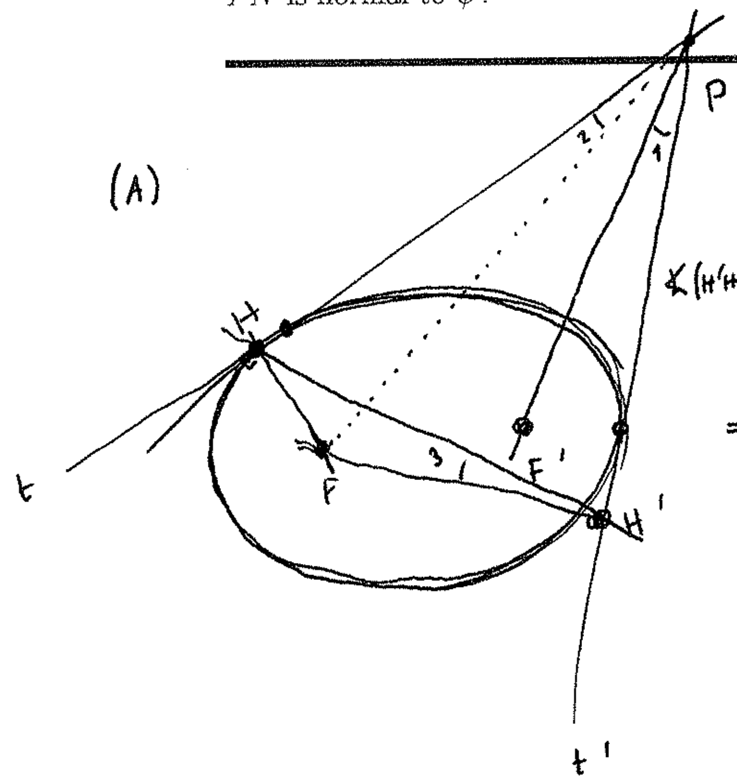
(B) From similar formulae and $|OI|^2 = R^2 - 2Rr$, it is found that

$|OI_a|^2 + |OI|^2 = |OI_b|^2 + |OI_c|^2$ iff $r_a - r = r_b + r_c$

which holds iff $a^2 = b^2 + c^2$!

4. (A) Let φ be an ellipse with foci $F\ F'$. Let t, t' be the tangents to φ which intersect in P. If H, H' are respectively the feet of the perpendiculars from F on t, t' prove that PF' is perpendicular to HH'.

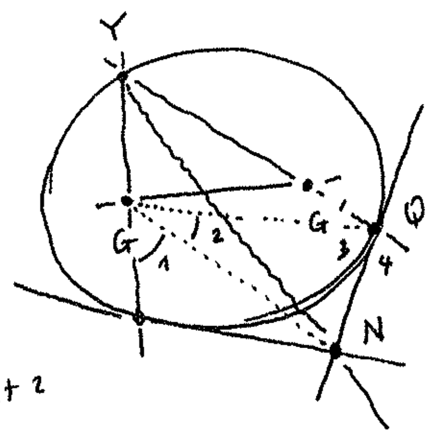
(B) Let ψ be an ellipse of foci $G\ G'$. For any $Y \in \psi$ choose $P, Q \in \psi$ such that $G \in YP$ and $G' \in YQ$. Let the tangents to ψ at P, Q intersect in N. Prove that YN is normal to ψ.



$\measuredangle(H'H, H'F) = $ "1" $\underset{\text{Poncelet 1}}{=}$ "2" $\underset{P,H,F,H' \text{ concyclic}}{=}$ "3". As $t' \perp FH'$ it follows that $PF' \perp HH'$!

"2" $= \measuredangle(PH, PF)$

"3" $= \measuredangle(PF', t')$ (mod π)



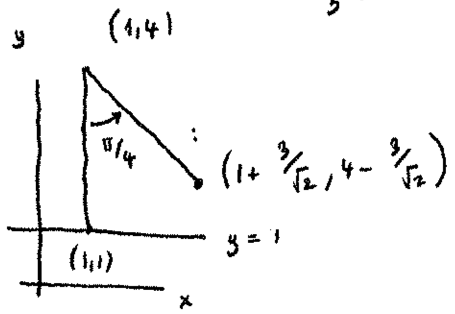
(B) Consider the triangle YGQ and notice that GN is an exterior bisector as $\underbrace{\text{"1} = \text{2"}}_{\text{Poncelet 2}}$ and QN is an exterior bisector as "3" $\underset{\text{reflection principle}}{=}$ "4". It follows that YN is an internal bisector.

5. (A) What is the image of the line $3x - 8y = 2$ under TR_u where $u = [-7, 5]$?
(B) What is the image of the line $y = 1$ under $\mathrm{Rot}_{(1,4),\pi/4}$?
(C) Identify $\mathrm{Rot}_{(1,2),\pi/2} \circ \mathrm{Rot}_{(2,3),3\pi/2}$.
(D) What is the image of the circle $x^2 + y^2 - 2x - 8 = 0$ under $\mathrm{HT}_{(5,1)}$?
(E) What is the image of the line $x + 2y - 5 = 0$ under $\mathrm{Hom}_{(2,1),7}$?

(A) $(2/3, 0)$ is on the line in question. Therefore $\left(\frac{2}{3} - 7, 0 + 5\right) = \left(-\frac{19}{3}, 5\right)$ is on the image which must be

$$3\left(x + \frac{19}{3}\right) - 8(y - 5) = 0 \quad \text{or} \quad \boxed{3x - 8y = -59}$$

(B)

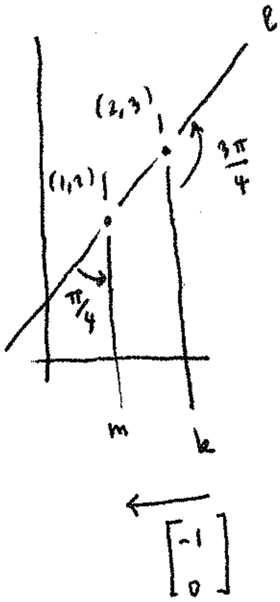


The image goes through $\left(1 + \frac{3}{\sqrt{2}}, 4 - \frac{3}{\sqrt{2}}\right)$ and has slope 1. It is

$$y - \left(4 - \frac{3}{\sqrt{2}}\right) = x - \left(1 + \frac{3}{\sqrt{2}}\right)$$

or $\boxed{y = x + 3 - 3\sqrt{2}}$

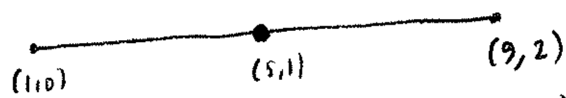
(C)



$$\mathrm{Rot}_{(1,2),\pi/2} \circ \mathrm{Rot}_{(2,3),3\pi/2} = \mathrm{Ref}_m \circ \mathrm{Ref}_k = \boxed{\mathrm{Tr}_{\vec{u}}}$$

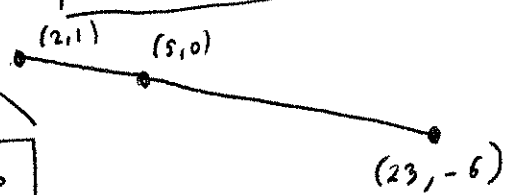
where $\vec{u} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

(D) $x^2 + y^2 - 2x - 8 = 0 \longrightarrow (x-1)^2 + y^2 = 9$



The image is the circle of center $(9,2)$, radius 3:

$$(x-9)^2 + (y-2)^2 = 9 \longrightarrow \boxed{x^2 + y^2 - 18x - 4y + 76 = 0}$$



(E) $(5,0)$ is on the line. Therefore the image must contain $(23,-6)$. The image is the line

$$(x - 23) + 2(y + 6) = 0 \quad \text{or} \quad \boxed{x + 2y - 11 = 0}$$