MATH 373 - GEOMETRY I

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

27th October 2014. Duration : 100 minutes. Three questions : 10 + 10 + 15 , 20 + 10 , 15 + 5 + 5 + 10

Solutions

1. (A) Given a triangle ABC and $T \in [B, C]$, prove the Stewart Relation

$$ax^2 = pb^2 + qc^2 - apq$$

where $a = |BC|$, $b = |CA|$, $c = |AB|$, $x = |AT|$, $p = |BT|$, $q = |TC|$, by employing the cosine rule or otherwise.

(B) Let S be the point in which the incircle touches BC. Prove that $|BS| = s - b$.

Cancelled !

(C) Prove that

$$|AS|^2 = s\left[\frac{(b-c)^2}{a} + s - a\right].$$

(A), (B) standard.

(C) is incorrect. I apologize for the unfortunate error.

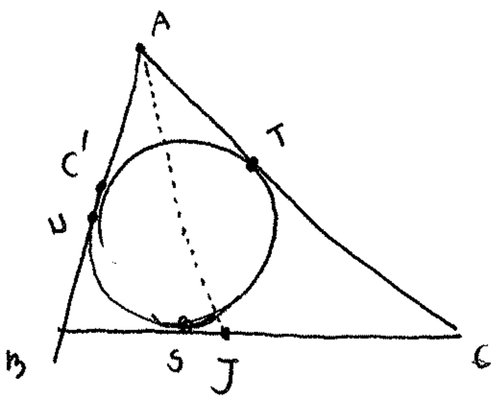
2. In a triangle ABC, let T be the point in which the incircle touches CA.

(A) Prove that BT and the median through C and the internal bisector through A are concurrent iff

$$b^2 + c^2 = (b+c)a$$

(B) Prove that in such a triangle $A \neq \pi/2$.

(A)



AJ, BT, CC' are concurrent (parallelity does not come into question!)

iff

$$-1 = \frac{JB}{JC} \cdot \frac{TC}{TA} \cdot \frac{C'A}{C'B}$$

$$= \left(-\frac{c}{b}\right)\left(-\frac{s-c}{s-a}\right)(-1)$$

which happens iff $c(s-c) = b(s-a)$

or equivalently $c(a+b-c) = b(b+c-a)$

or equivalently

$$b^2 + c^2 = a(b+c)$$

(B) $A = 90^\circ \longrightarrow \underbrace{b^2+c^2}_{= a^2} = (b+c)a \longrightarrow a = b+c$ impossible unless the triangle is degenerate...

3. In a triangle ABC, let k, l, m be the perpendicular bisectors of $[B,C]$, $[C,A]$, $[A,B]$. Consider $U \in k$, $V \in l$, $W \in m$ that satisfy the conditions $C \in UV$, $A \in VW$, $B \in WU$.

(A) Show that the angle $< BCU$ is equal to A.

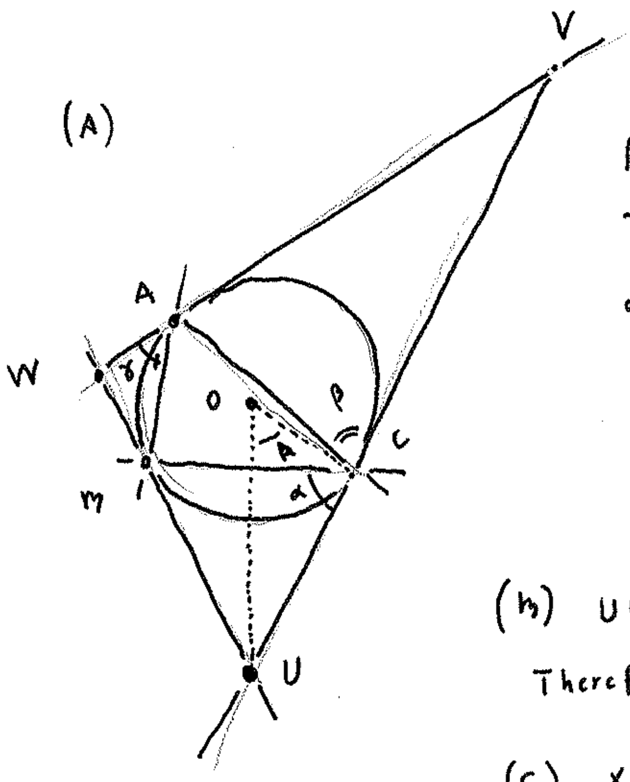
(B) Where is the incenter of UVW?

(C) What is the incircle of UVW?

(D) Prove that

$$\Delta_{UVW} = R^2\Big(\tan A + \tan B + \tan C\Big)$$

(A)



$$\beta = \pi - \alpha - C$$

$$\gamma = \pi - \beta - A = \alpha + C - A$$

$$\alpha = \pi - \gamma - B = \pi - \alpha - B - C + A$$

$$= -\alpha + 2A$$

$$\longrightarrow \quad \underline{\alpha = A}$$

(B) UO, VO, WO are internal angle bisector. Therefore O is the incenter of UVW

(C) $\measuredangle BCU = A$ implies that the circumcircle of ABC is tangent to UV. Similar reasoning with other sides shows that the circumcircle of ABC is is incircle of UVW.

(D) In the right triangle OUC : $|UC| = R \tan A$

Hence $|UV| = R\,(\tan A + \tan B)$.

$$\left.\begin{array}{l}\text{The semiperimeter of } UVW \text{ is } R(\tan A + \tan B + \tan C). \\ \text{The inradius of } UVW \text{ is } R.\end{array}\right\} \checkmark$$