MATH 373 - GEOMETRY I

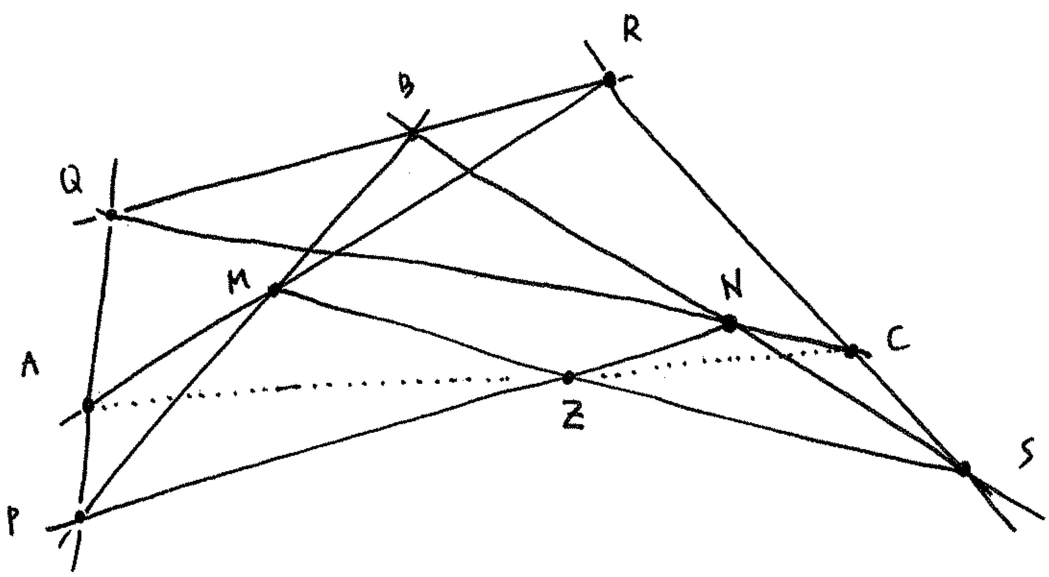
SECOND MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

3rd December 2014. Duration : 100 minutes. Three questions : 20 , (10 + 5) + 10 + (10 + 5) , 5 + 10 + 15 + 10

Solutions

1. Given distinct non-collinear points P, Q, R, S consider points $A \in PQ - \{P, Q\}$, $B \in QR - \{Q, R\}$, $C \in RS - \{R, S\}$. Let PB and AR, QC and BS meet in M, N, respectively. Finally, let PN intersect MS in Z. By employing the theorem of Desargues or otherwise prove that A, Z, C are collinear.



Note that $PA \cap NC = \{Q\}$, $AM \cap CS = \{R\}$, $MP \cap SN = \{B\}$ and the points Q, B, R are collinear. By the Desargues theorem applied to the triangles PAM and NCS it follows that PN, AC, MS are concurrent or parallel. As $PN \cap SM = \{Z\}$ it is seen that A, Z, C are collinear...

nonconcentric

2. (A) Compute the power of the point $\Omega(p,q)$ with respect to the circle $x^2+y^2+2ax+2by+c=0$. Write down the equation of the radical axis of the circles $x^2+y^2+2a_1x+2b_1y+c_1=0$ and $x^2+y^2+2a_2x+2b_2y+c_2=0$.

(B) Consider fixed non-concentric circles Γ_1, Γ_2. Prove that the set of points whereof the powers with respect to Γ_1 and Γ_2 differ by a constant is a line parallel to the radical axis of Γ_1 and Γ_2.

(C) Prove that the set of points whereof the ratio of the power with respect to Γ_1 to the power with respect to Γ_2 equals a constant $\lambda \neq 1$ is a circle. Find the center and the radius of this circle if Γ_1 is the circle of center $(3,7)$ and radius 4 and Γ_2 is the circle of center $(1,3)$ and radius 5 and $\lambda = 2$.

(A) The distance of $\Omega(p,q)$ from the center $O(-a,-b)$ is $(p+a)^2+(q+b)^2$.
The square of the diameter r^2 is a^2+b^2-c. Hence the power of Ω is

$$|\Omega O|^2 - r^2 = (p+a)^2+(q+b)^2-(a^2+b^2+c) = p^2+q^2+2ap+2bq+c.$$

Hence (x,y) has equal powers w.r.to the given circles iff

$$x^2+y^2+2a_1x+2b_1y+c_1 = x^2+y^2+2a_2x+2b_2y+c_2$$

or

$$\boxed{2(a_1-a_2)x+2(b_1-b_1)y+c_1-c_2=0}$$

(B) The same reasoning:

$$(x^2+y^2+2a_1x+2b_1y+c_1)-(x^2+y^2+2a_2x+2b_2y+c_2)=\overset{\text{constant}}{K}$$

$$\longrightarrow \boxed{2(a_1-a_2)x+2(b_1-b_2)y+c_1-c_2-K=0}$$

a line parallel to the radical axis...

(C) Similarly

$$\frac{x^2+y^2+2a_1x+2b_1y+c_1}{x^2+y^2+2a_2x+2b_2y+c_2}=\lambda \longrightarrow \left[x^2+y^2+2\overbrace{\frac{a_1-\lambda a_2}{1-\lambda}}^{a_3}x+2\overbrace{\frac{b_1-\lambda b_2}{1-\lambda}}^{b_3}y+\underbrace{\frac{c_1-\lambda c_2}{1-\lambda}}_{c_3}=0\right]$$

a circle!

$\Gamma_1 \longrightarrow a_1=-3$, $b_1=-7$, $r_1=4$

$\Gamma_2 \longrightarrow a_2=-1$, $b_2=-3$, $r_2=5$

$$\frac{x^2+y^2-6x-14y+42}{x^2+y^2-2x-6y-15}=2 \longrightarrow x^2+y^2+2x+2y-72=0$$

$$\boxed{\begin{array}{ll}\text{Center} & (-1,-1)\\ \text{Radius} & \sqrt{74}\end{array}}$$

3. (A) Prove that for any rectangle $ABCD$

$$|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2$$

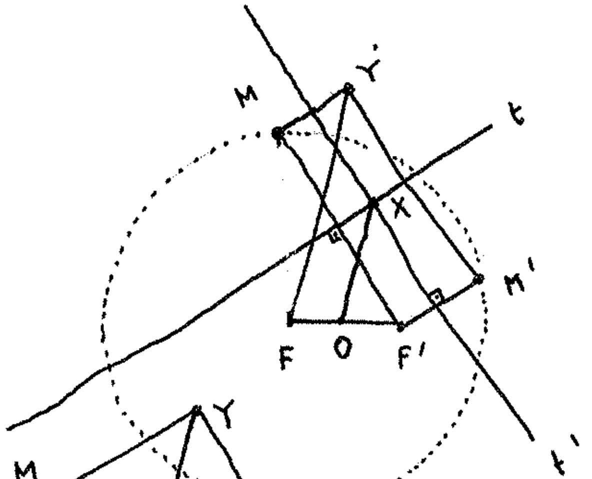
holds for any point P .

(B) Let φ be an ellipse. Prove that the set of points from which the tangents to φ are perpendicular to one another, is a circle.

(C) Let φ, ψ be ellipses with common foci. Prove that the set of points from which the tangents to φ and to ψ are perpendicular to each other, is a circle.

(D) What can you say about the set of points from which the tangents to a parabola are perpendicular to one another ?

(A) ✓ (B)



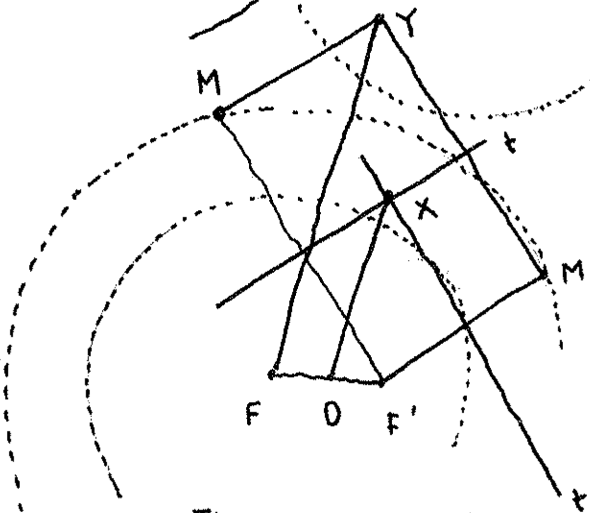
Let O be the midpoint of $[F, F']$.

$$|OX|^2 = \frac{1}{4}|FY|^2$$

$$= \frac{1}{4}\left[|FM|^2 + |FM'|^2 - |FF'|^2\right]$$

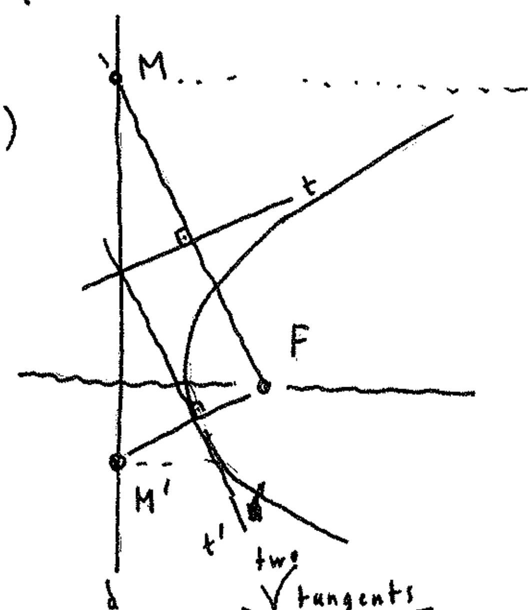
$=$ constant. Hence ...

(C)



The same argument...

(D)



In a parabola two tangents perpendicular to one another intersect on the directrix...

The details...