Department of Mathematics, METU, MATH 373 - GEOMETRY-I

FINAL EXAMINATION

STUDENT NUMBER	NAME, FAMILY NAME	GRADE

18th January 2016. Duration : 125 minutes.
Five questions : 5 + 8 + 7 , ~~8 + 8 + 9~~ , 6 + 7 + 7 , 10 + 10 , 5 + 5 + 5 + 5
10 + 10

Solutions

1. Let Δ_{ABC} denote the area of the triangle ABC.

(A) Prove that $\Delta_{ABC} = sr = (s-a)\,r_a$.

(B) For any $\lambda \neq -1$ prove that $r_b + r_c = \lambda(r_a - r)$ iff

$$\cos A = \frac{\lambda - 1}{\lambda + 1} \quad .$$

(C) Compute the angle A in a triangle ABC in which

$$3r + r_b + r_c = 3r_a \ .$$

(A) Standard. $\longrightarrow$ 1998 MT 1 (A)

(B) $r_b + r_c = \lambda\,(r_a - r)$

$$\longrightarrow \frac{1}{s-b} + \frac{1}{s-c} = \lambda\left[\frac{1}{s-a} - \frac{1}{s}\right] \longrightarrow \lambda = \frac{s(s-a)}{(s-b)(s-c)}$$

$$= \frac{(b+c)^2 - a^2}{a^2 - (b-c)^2} \longrightarrow \text{routine by the Cosine Rule.}$$

(C) 2013 MT 1 2 (B)

or $\quad \cos A = \dfrac{3-1}{3+1} = \dfrac{1}{2} \longrightarrow A = \pi/3$.

2. Given a triangle ABC, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that AP, BQ, CR are concurrent. Let QR, RP, PQ meet BC, CA, AB in X, Y, Z respectively. Prove that

(A) X, Y, Z are collinear.

(B) AP, BY, CZ are concurrent or parallel.

2010 Final, Question 1

3. Given a triangle ABC, let α, β, γ be the circles of respective diameters $[B, C]$, $[C, A]$, $[A, B]$.

(A) Prove that the radical axis of β and γ is the altitude through A.

(B) Prove that the radical center of α, β, γ is the orthocenter H of ABC.

(C) Show that the power of H with respect to any of α, β, γ equals $-4R^2 \cos A \cos B \cos C$.

2013 Final, Question 3

4. Let φ be an ellipse of foci F, F'. Let $\varphi \cap FF' = \{A, A'\}$. Let ℓ, ℓ' be the tangents to φ. at A, A'. For any tangent t to φ at $M \in \varphi$ let $t \cap \ell = \{P\}, t \cap \ell' = \{P'\}$.

(A) Prove that

$$< (FP, FP') =< (F'P, F'P') = \pi/2$$

(B) Prove that $FP, F'P'$ intersect on the normal to φ at M.

2013 Final, Question 4

5. (A) What is the image of the line $2x + 3y = 7$ under TR_{u} where $\mathrm{u} = [-1, 3]$?

(B) Let P, Q be distinct points. Prove that half-turns in P and Q together give rise to a translation. Which ?

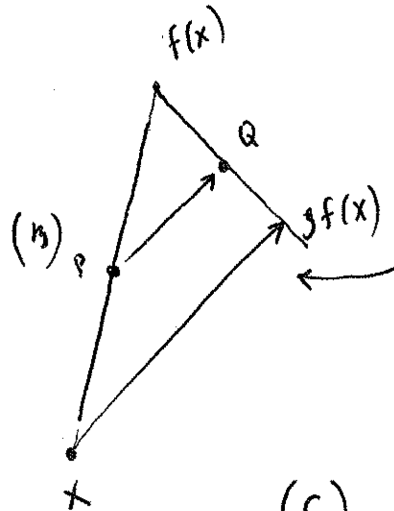
(C) What is the image of the circle $x^2 + y^2 - 2y - 3 = 0$ under $\mathrm{Hom}_{(2,1),3}$?

(D) What is the image of the line $y = 0$ under $\mathrm{Inv}_{(2,3),6}$?

(A) $$\mathrm{TR}_{[-1,3]}\{(x,y) \mid 2x+3y=7\} = \{(x-1, y+3) \mid 2x+3y=7\}$$

$$= \{(x',y') \mid 2(x'+1) + 3(y'-3) = 7\}$$

$$= \{(x',y') \mid \boxed{2x'+3y' = 14}\}$$

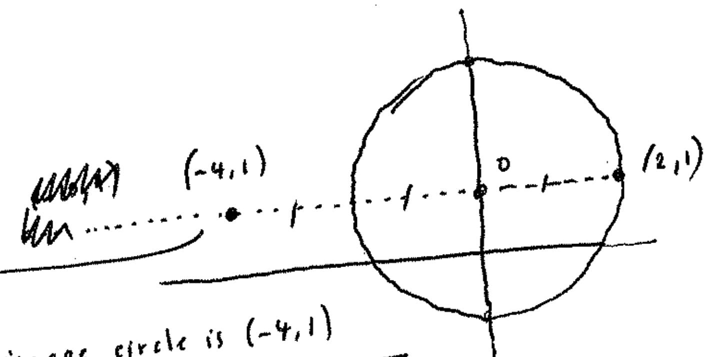
(B) 

Put $f = HT_P$ & $g = HT_Q$

$$HT_Q \circ HT_P = TR_{\overrightarrow{PQ}}.$$

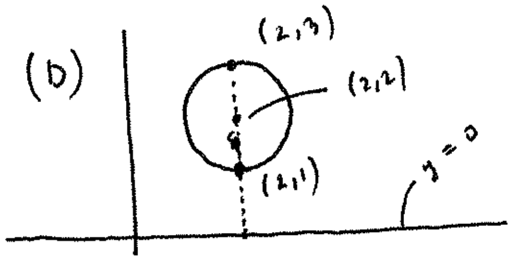
(C) $x^2+y^2-2y-3 = 0$

$x^2+(y-1)^2 = 4$



The center of the image circle is $(-4,1)$

The radius is 6. $\boxed{(x+4)^2+(y-1)^2 = 36}$

(D) 

The image is the circle $(x-2)^2+(y-2)^2 = 1$