Department of Mathematics, METU, MATH 373 - GEOMETRY-I

FIRST MIDTERM

STUDENT NUMBER	NAME, FAMILY NAME	GRADE

13th November 2015. Duration : 90 minutes. Three questions : 15 + 20 , 15 + 20 , 10 + 20

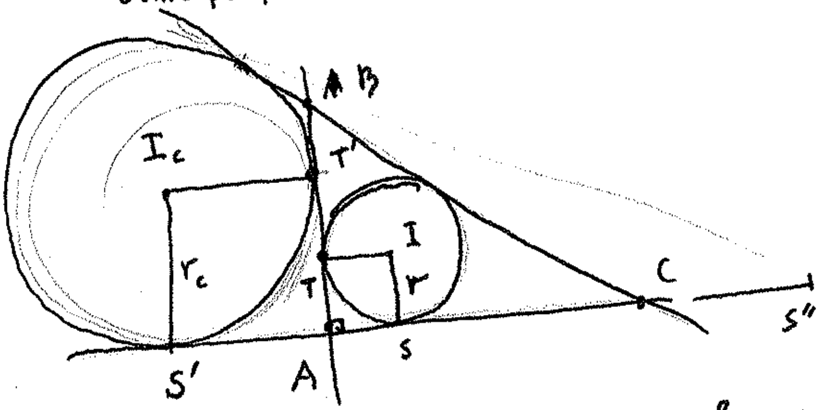
Solutions

1. (A) In a triangle ABC prove that $A = 90^\circ$ iff $r + r_b + r_c = r_a$.

(B) Let P be the point on the circumcircle of a triangle ABC such that AP is the internal bisector of the angle A. Prove that $|PA| = 2|PB|$ iff $2\sin\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$.

(A) The standard solution $\longrightarrow$ MT1 2013, Question 2(A).

Some people note the following elegant solution: (My congratulations!)

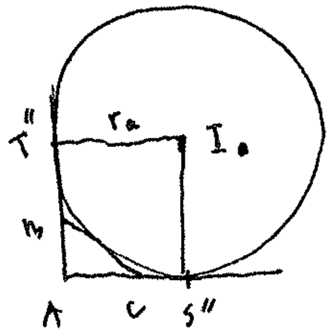


$A = 90^\circ$ iff $ITAS$ and $I_cS'AT'$, $I_aT''AS''$ are squares...

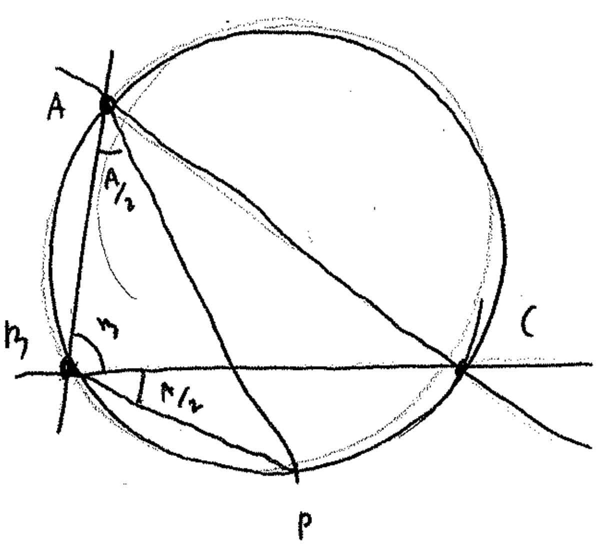
Equivalently: $r = |AT| = s-a$

& $r_c = |AT'| = s-b$ & $r_b = s-c$

$r_a = |AS''| = s$. Hence...



(B)



The sine rule in ABP:

$$\frac{|PB|}{\sin A/2} = \frac{|PA|}{\sin\left(\frac{A}{2} + B\right)}$$

Hence $|PA| = 2|PB|$ iff $\pi - B - C$

$$2\sin\left(\frac{A}{2}\right) = \sin\left(\frac{A}{2} + B\right)$$

$$= \cos\left(\frac{B-C}{2}\right)$$

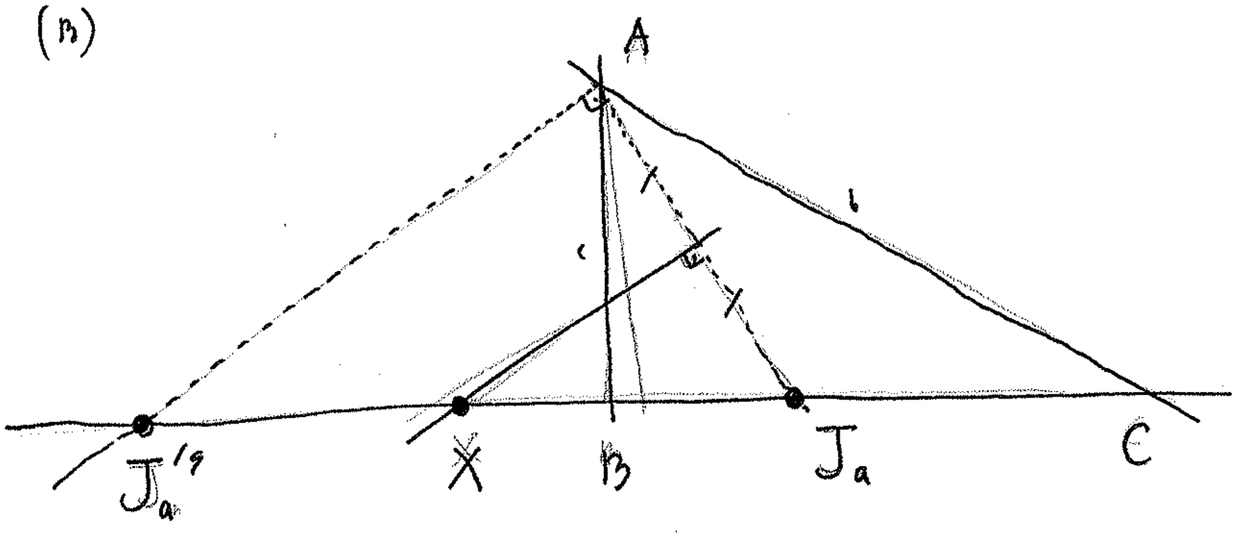
2. In a triangle ABC, let $J_a \in BC$, $J_b \in CA$, $J_c \in AB$ be chosen such that AJ_a, BJ_b, CJ_c be the respective internal angle bisectors at A, B, C.

(A) Prove that AJ_a, BJ_b and CJ_c are concurrent.

(B) Let the perpendicular bisectors of $[A, J_a]$, $[B, J_b]$, $[C, J_c]$ intersect BC, CA, AB in X, Y, Z respectively. Prove that X, Y, Z are collinear.

(A) Routine

(B)



Simple computations result in (assume $b > c$). $|J_a B| = \frac{ca}{b+c}$

$|J_a' B| = \frac{ca}{b-c}$ & $|J_a J_a'| = \frac{2abc}{b^2-c^2}$, $|XB| = \frac{ac^2}{b^2-c^2}$

$|XC| = \frac{ab^2}{b^2-c^2}$ hence

signed!

$$\frac{XB}{XC} = \frac{c^2}{b^2}.$$ Similarly with $\frac{YC}{YA}$, $\frac{ZA}{ZB}$

$\longrightarrow$ Menelaus.

There are people who obtain this in a much more elegant fashion. I congratulate them...

3. (A) State and prove the Stewart relation in a triangle.

(B) Given a non-isosceles triangle ABC compute the length n_a' of the external angle bisector through A. in terms of the side lengths a, b, c. (To be pricise, $n_a' = |AJ_a'|$ where J_a' is the point in which the external angle bisector through A intersects BC.)

(A) Routine

(B) Lecture notes, p. 13.