Department of Mathematics, METU, MATH 373 - GEOMETRY-I

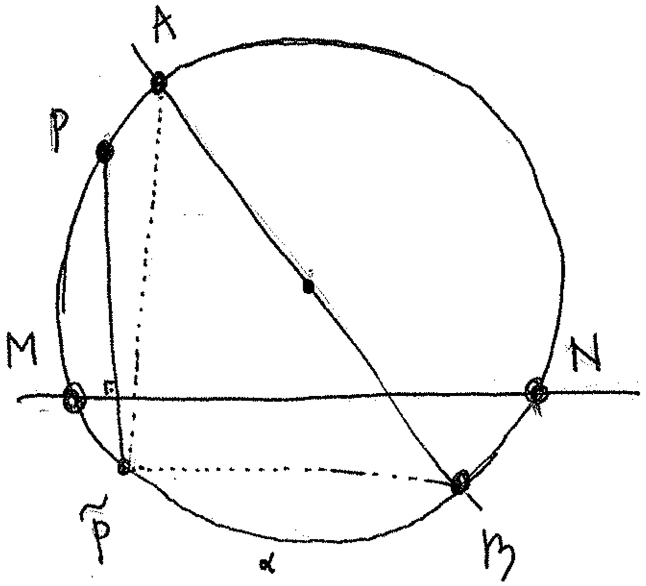
SECOND MIDTERM

STUDENT NUMBER	NAME, FAMILY NAME	GRADE

18th December 2015. Duration : 90 minutes. Three questions : 30 , 20 + 20 , 15 + 15

Solutions

1. Given a circle α, consider points $P, M, N \in \alpha$ and let $[A, B]$ be a diameter of α. Prove that the Simson line of P with respect to the triangle AMN is perpendicular to the Simson line of P with respect to the triangle BMN.



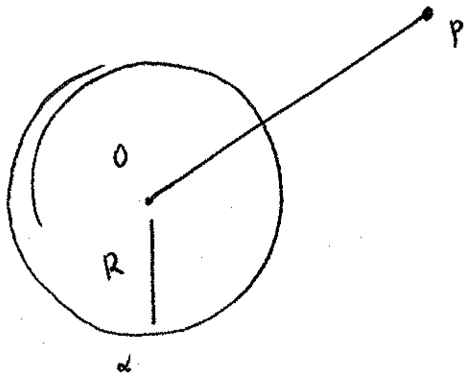
Let the perpendicular from P onto MN intersect α in P, $\tilde{P}$. The Simson line k of P w.r.to AMN is parallel to $\tilde{P}A$ whereas the Simson line ℓ of P w.r.to BMN is parallel to $\tilde{P}B$. It follows that

$$\measuredangle(k, \ell) = \measuredangle(\tilde{P}A, \tilde{P}B) = \pi/2 \pmod{\pi}$$

2. (A) Characterise the set of points whereof the power with respect to a fixed circle is a constant.

(B) Consider circles γ and δ which are tangent to the line k at the points $C, D \in k$ respectively. Prove that the radical axis of γ and δ bisects the line segment $[C, D]$.

(A) Let α be the circle in question. Let its center be O and its radius R.

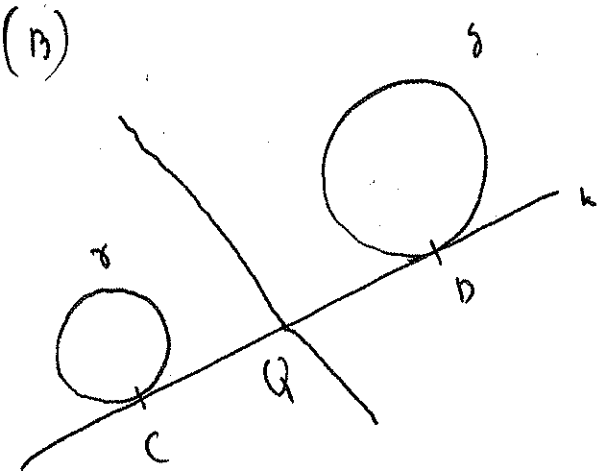


~~The~~ A point P with power C with respect to α satisfies the equation

$$|PO|^2 - R^2 = C.$$

It follows that the points with power C with respect to α constitute a circle of center O and radius $\sqrt{C+R^2}$.

(B)



Let the radical axis of γ, δ intersect k in Q.

The power of Q w.r.to γ $= |QC|^2$, w.r.to δ $= |QD|^2$. Hence $|QC| = |QD|$.

It follows that Q is the midpoint of $[C, D]$.

3. (A) Let ψ be an ellipse. Prove that the set of points from which the tangents to ψ are perpendicular to one another, is a circle. (*Hint* : Remember that for any rectangle $ABCD$

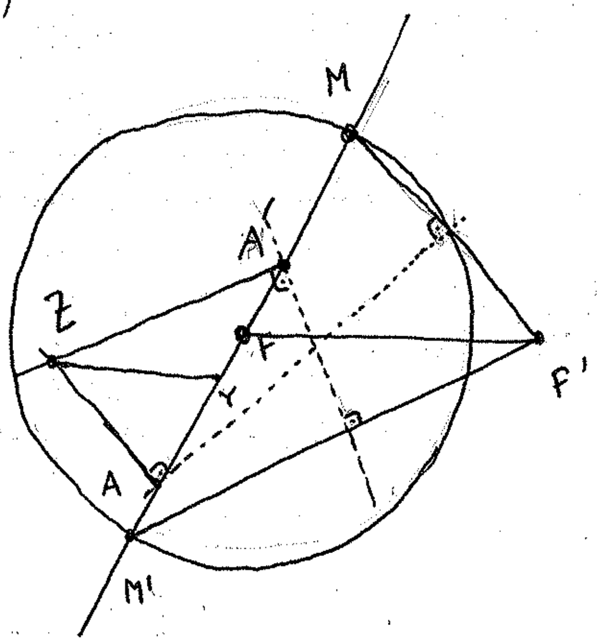
$$|PA|^2 + |PC|^2 = |PB|^2 + |PD|^2$$

holds for any point P .

(B) Consider a hyperbola φ with foci F, F'. Let a line through F meet φ in A, A'. If the normals to φ at A, A' intersect in Z, prove that the parallel to FF' through Z bisects $[AA']$.

(A) Set already in the final examination in 2011.

(B)



The perpendicular bisectors of $[M,F']$ and $[M,F]$ are tangents, the perpendiculars thereto at A' and A are normals. Hence

$$A'Z \parallel M'F'$$
$$AZ \parallel MF'$$

hence $AA'Z$ and $MF'M'$ have parallel sides. FF' is the median of $MF'M'$ and consequently the median ZY in AZA' must be parallel to FF' !