MATH 373 - GEOMETRY I

FINAL EXAMINATION

FAMILY NAME	OTHER NAMES	GRADE

11th January 2011. Duration : 2.5 hours.

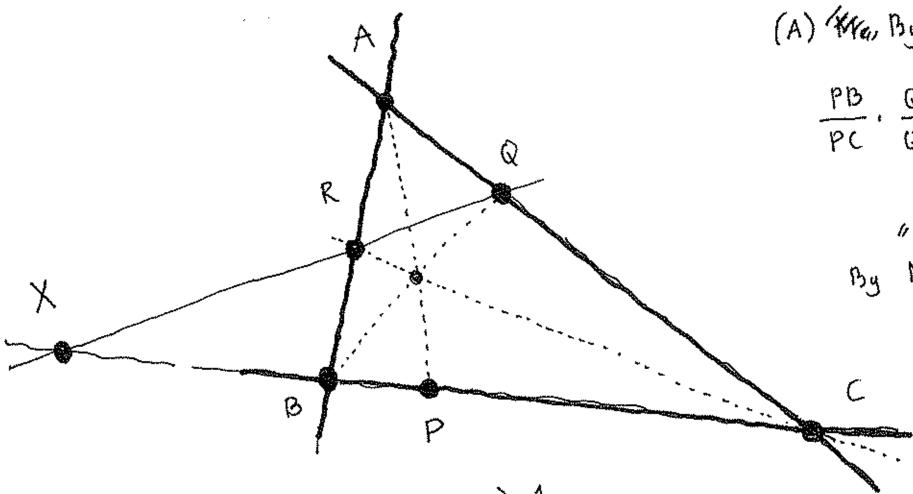
Six questions : $8+8$, $4+8+4+4+4$, $8+4+8$, 8 , 8 , $4+4+4+4+8$

Solutions

1. Given a triangle ABC, consider $P \in BC - \{B, C\}$, $Q \in CA - \{C, A\}$, $R \in AB - \{A, B\}$ such that AP, BQ, CR are concurrent. Let QR, RP, PQ meet BC, CA, AB in X, Y, Z respectively. Prove that

(A) X, Y, Z are collinear.

(B) AP, BY, CZ are concurrent or parallel.



(A) ~~Hen~~ By "Ceva" in ABC

$$\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB} = -1$$ since AP, BQ, CR are concurrent.

By "Menelaus" in ABC with QR:

$$\frac{XB}{XC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB} = +1$$

hence $\frac{XB}{XC} = \left(\frac{QC}{QA} \cdot \frac{RA}{RB}\right)^{-1}$. We can similarly compute $\frac{YC}{YA}$ and $\frac{ZA}{ZB}$ and

$$\frac{XB}{XC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \bigg(\underbrace{\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB}}_{=-1}\bigg)^{-2} = 1$$. Consequently by "Menelaus", X, Y, Z are collinear.

(B) As $$\frac{PB}{PC} \cdot \frac{YC}{YA} \cdot \frac{ZA}{ZB} = \frac{PB}{PC}\left(\frac{RA}{RB}\,\frac{PB}{PC}\right)^{-1}\left(\frac{PB}{PC} \cdot \frac{QC}{QA}\right)^{-1}$$

$$= \bigg(\underbrace{\frac{PB}{PC} \cdot \frac{QC}{QA} \cdot \frac{RA}{RB}}_{=-1}\bigg)^{-1} = -1$$

we conclude by "Ceva", that AP, BY, CZ are concurrent or parallel.

2. Let ABC be a triangle with orthocenter H, incenter I and centroid G. Given a point X, let G_a, G_b, G_c be the centroids of XBC, XCA, XAB respectively.

(A) Prove that G_bG_c is parallel to BC and GG_a is parallel to AX .

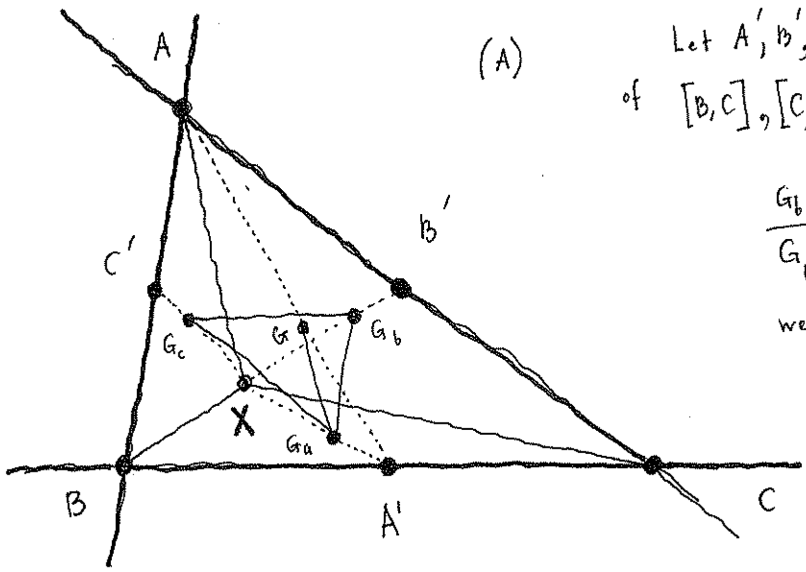
(B) How can we choose $\alpha, \beta \in \mathbb{R}$ so that

$$\mathrm{Hom}\Big(X,\beta\Big) \circ \mathrm{Hom}\Big(G,\alpha\Big)(ABC) = G_aG_bG_c \ ?$$

(C) Show that the centroid of $G_aG_bG_c$ is on the line XG . Where is it exactly ?

(D) If $X = H$ prove that G is the orthocenter of the triangle $G_aG_bG_c$.

(E) If $X = I$ which remarkable point of the triangle $G_aG_bG_c$ is the point G ?



(A) Let A', B', C' be the respective midpoints of $[B,C]$, $[C,A]$, $[A,B]$. Since

$$\frac{G_bX}{G_bB'} = \frac{G_cX}{G_cC'} = -\frac{1}{2}$$

we find that

$$G_bG_c \parallel B'C' \parallel BC .$$

Similarly $\frac{GA'}{GA} = \frac{G_aA'}{G_aX} = -\frac{1}{2}$

hence $GG_a \parallel XA$

(B) Since $\mathrm{Hom}(X, \frac{2}{3}) : \begin{cases} A' \longrightarrow G_a \\ B' \longrightarrow G_b \\ C' \longrightarrow G_c \end{cases}$. Clearly $\alpha = -\frac{1}{2}$, $\beta = \frac{2}{3}$

(C) The centroid of $G_aG_bG_c$ is $Z = \mathrm{Hom}(X, \frac{2}{3}) \circ \mathrm{Hom}(G, -\frac{1}{2})(G)$
$= \mathrm{Hom}(X, \frac{2}{3})(G)$. $\therefore$ Z is on XG .
(and indeed $\frac{ZX}{GX} = \frac{2}{3}$)

(D) If $X = H$, then $G_bG_c \parallel BC \perp AH \parallel GG_a$.
Similarly $GG_b \perp G_cG_a$. $\therefore$ G is the orthocenter of $G_aG_bG_c$.

(E) AI bisects the angle $\measuredangle BAC$ internally. As $CA \parallel G_cG_a$, $BA \parallel G_bG_a$ and $IA \parallel GG_a$ we conclude that G_aG bisects the angle $\measuredangle G_bG_aG_c$ internally. Similarly ...
Therefore G is the incenter of $G_aG_bG_c$.

3. Consider a positively oriented triangle ABC and positively oriented equilateral triangles BSC, CTA, AUB.

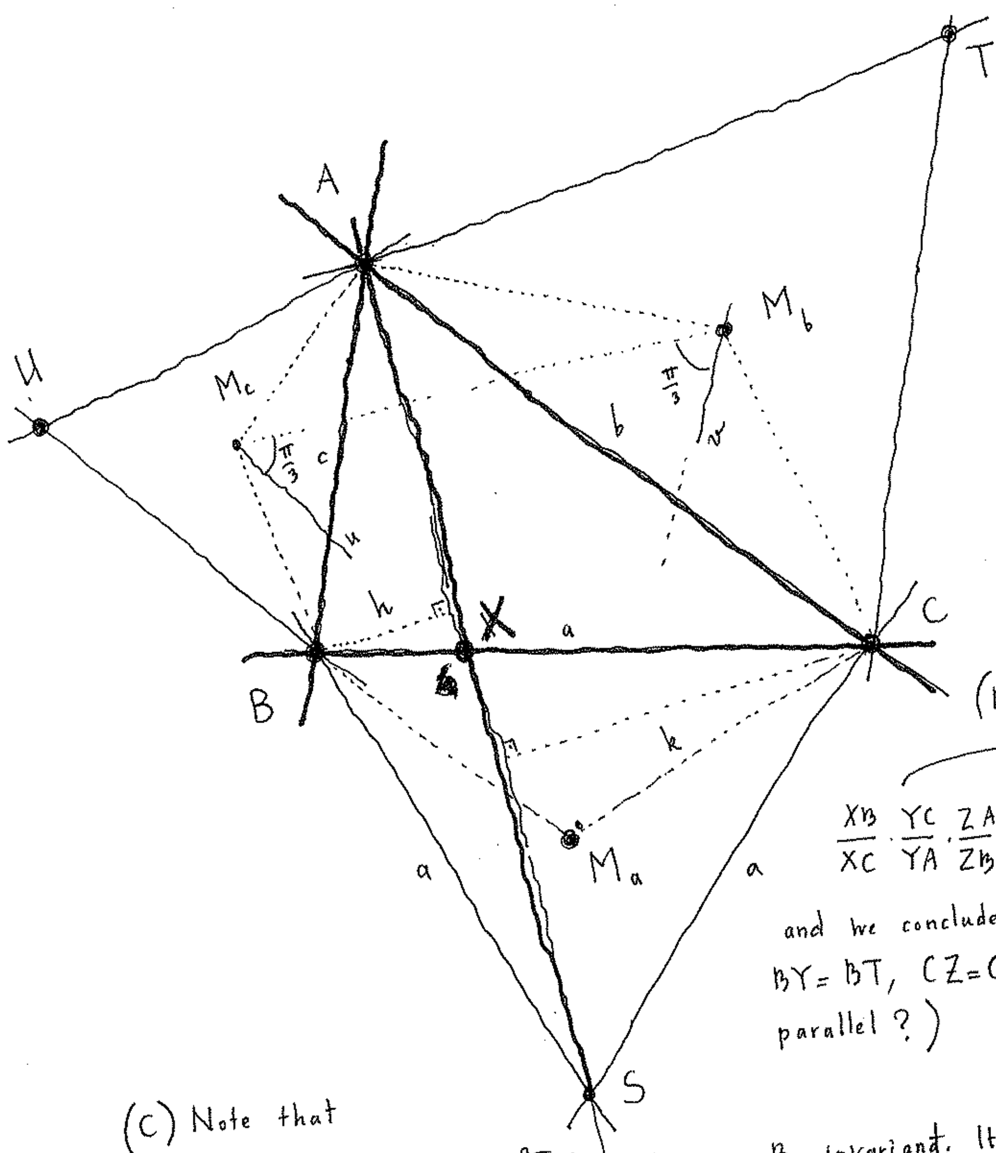
(A) Let AS intersect BC in X. Comparing the (oriented) areas of the triangles $A\overset{S}{\cancel{P}}B$ and $A\overset{S}{\cancel{P}}C$ compute XB/XC in terms of the angles B and C.

S (B) Prove that AS, BT, CU are concurrent.

(C) Let M_a, M_b, M_c be the centers of the equilateral triangles BSC, CTA, AUB. First proving

$$\mathrm{Rot}\Big(M_b, 2\pi/3\Big) \circ \mathrm{Rot}\Big(M_c, 2\pi/3\Big) = \mathrm{Rot}\Big(M_a, -2\pi/3\Big)$$

or otherwise, show that $M_aM_bM_c$ is an equilateral triangle.



(A) $$\frac{XB}{XC} = -\frac{h}{k} = \frac{\Delta_{ABS}}{\Delta_{ACS}} = \frac{ac\sin(B+\pi/3)}{-a\cdot b\sin(C+\pi/3)}$$

$$= -\frac{c\left(\sin B\,\frac{1}{2} + \cos B\cdot\frac{\sqrt{3}}{2}\right)}{b\left(\sin C\,\frac{1}{2} + \cos C\,\frac{\sqrt{3}}{2}\right)} = \boxed{-\frac{1+\sqrt{3}\cot B}{1+\sqrt{3}\cot C}}$$

(B)

$$\frac{XB}{XC}\cdot\frac{YC}{YA}\cdot\frac{ZA}{ZB} = \left(-\frac{1+\sqrt{3}\cot B}{1+\sqrt{3}\cot C}\right)\left(\dots\right)\left(\dots\right) = -1.$$

and we conclude by "Ceva" that $AX = AS$, $BY = BT$, $CZ = CU$ are concurrent. (Why not parallel?)

(C) Note that

$\mathrm{Rot}(M_a, 2\pi/3)\circ\mathrm{Rot}(M_b, 2\pi/3)\circ\mathrm{Rot}(M_c, 2\pi/3)$ leaves B invariant. It is also a translation since $2\pi/3 + 2\pi/3 + 2\pi/3 = 0 \pmod{2\pi}$. $\therefore$ It is the identity map $\longrightarrow$ The desired identity.

It follows that the line u through M_c and v through M_b with $\measuredangle(u, M_bM_c) = \measuredangle(M_bM_c, v) = \pi/3$ intersect in M_a!

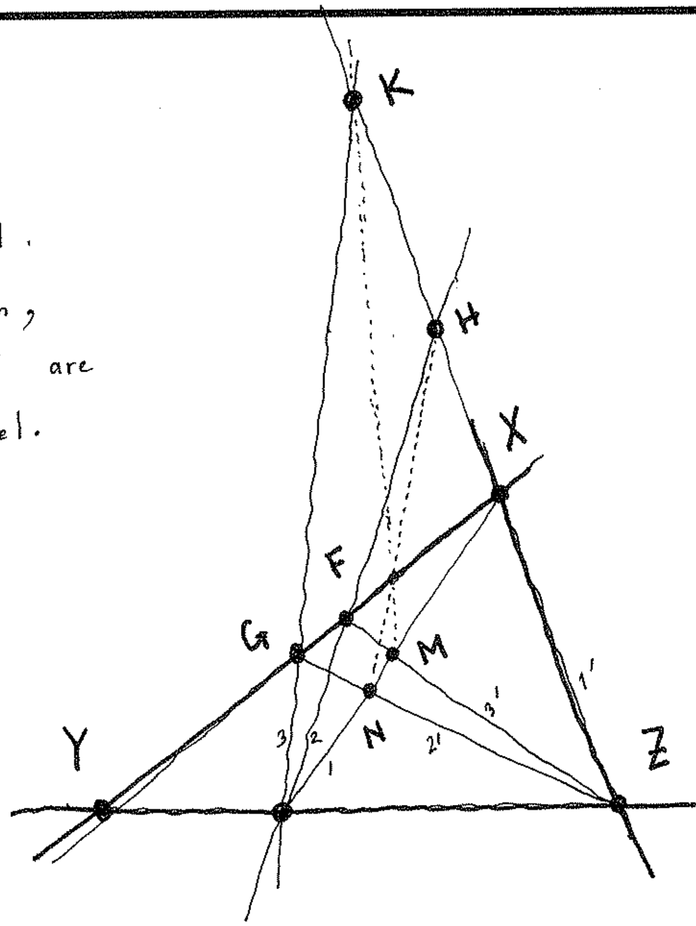
4. Given triangle XYZ, let T be a point on YZ and let F, G be points on XY. Let FT meet XZ in H, GT meet XZ in K, ZF meet XT in M and ZG meet XT in N. Prove that KM and HN intersect on on XY or KM, HN, XY are parallel.

Note: $1 \cdot 2' = N \quad 1' \cdot 2 = H$

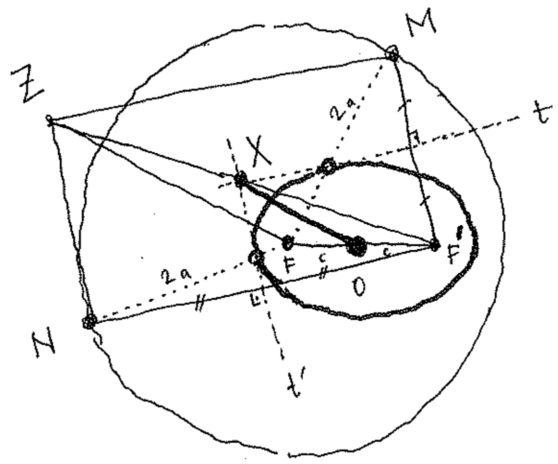
$2 \cdot 3' = F \quad 2' \cdot 3 = G$

$3 \cdot 1' = K \quad 3' \cdot 1 = M$.

By the dual of the Pappus theorem, HN, KM and $FG = XY$ are either concurrent ~~A~~ or parallel.



5. Let φ be an ellipse. Prove that the set of points from which the tangents to φ are perpendicular to one another, is a circle.



$$|OX|^2 = |ZF|^2 = |FM|^2 + |FN|^2 - |FF'|^2$$
$$= 8a^2 - 4c^2 = \text{constant !}$$

The midpoint of $[F, F']$

6. (A) What is the image of the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$ under the inversion $\mathrm{Inv}\Big((0,0),1\Big)$?

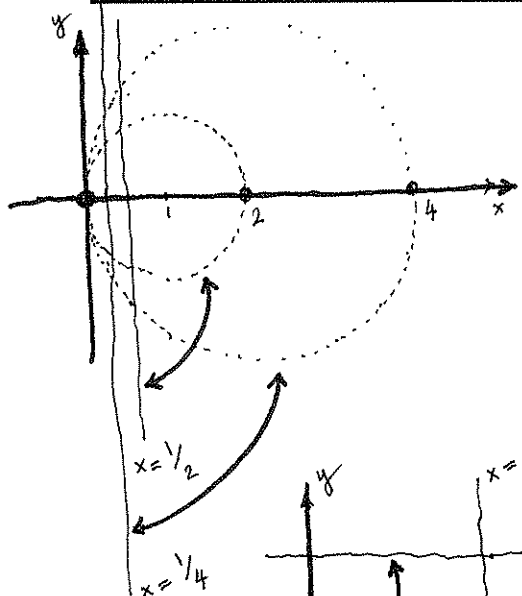
(B) What is the image of the lines $x = 3$ and $x = 5$ under the inversion $\mathrm{Inv}\Big((0,0),30\Big)$?

(C) What is the image of the circles $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ under the inversion $\mathrm{Inv}\Big((0,0),6\Big)$?

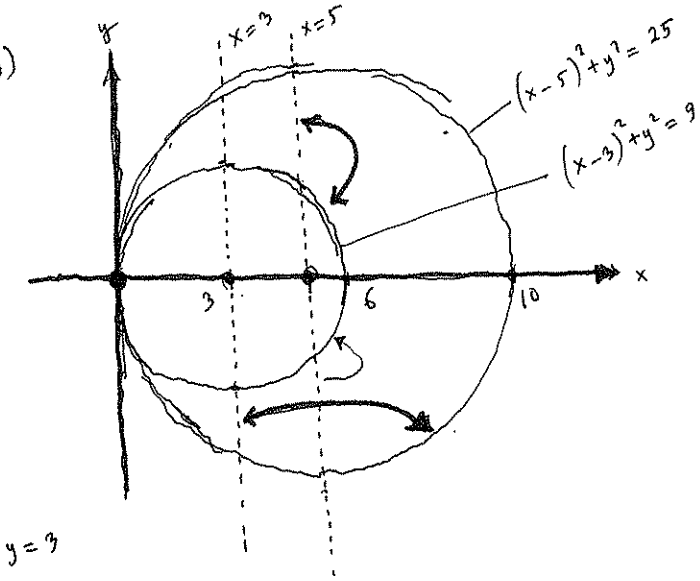
(D) What is the image of the circle $(x-1)^2 + y^2 = 1$ under the inversion $\mathrm{Inv}\Big((-1,0),6\Big)$?

(E) Find the homothety centers of the circles in (D).

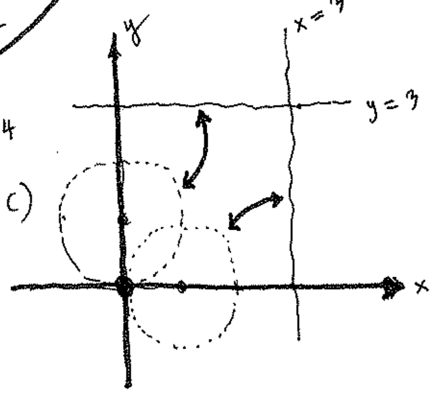
(A)



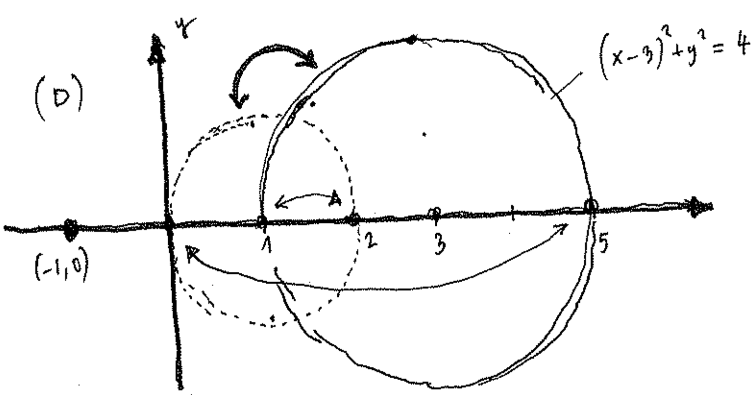
(B)



(C)



(D)



(E) One of the homothety centers (the external one!) is $(-1,0)$ since it is the inversion center. The other is $(x,0)$ with $= P_{ext}$

$$\frac{x-1}{x-3} = -\frac{1}{2} \longrightarrow 2x - 2 = -(x-3)$$

$$x = 5/3$$

$$\therefore \quad P_{int} = \left(5/3, 0\right)$$