MATH 373 - GEOMETRY I

FIRST MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

5th November 2010. Duration : 100 minutes. Three questions : 30 , 30 , 5 + 15 + 10 + 10

Solutions

1. In a triangle ABC, compute the angle B if

$$r_c r_a = \left(3 + 2\sqrt{2}\right) r\, r_b \quad .$$

$$r_c r_a = \left(3 + 2\sqrt{2}\right) r\, r_b$$

$$\frac{\Delta}{s-c} \frac{\Delta}{s-a} = \left(3 + 2\sqrt{2}\right) \frac{\Delta}{s} \frac{\Delta}{s-b}$$

$$s(s-b) = \left(3 + 2\sqrt{2}\right)(s-a)(s-c)$$

$$(a+b+c)(a+c-b) = \left(3 + 2\sqrt{2}\right)(b+c-a)(b+a-c)$$

$$(a+c)^2 - b^2 = \left(3 + 2\sqrt{2}\right)\left[b^2 - (a-c)^2\right]$$

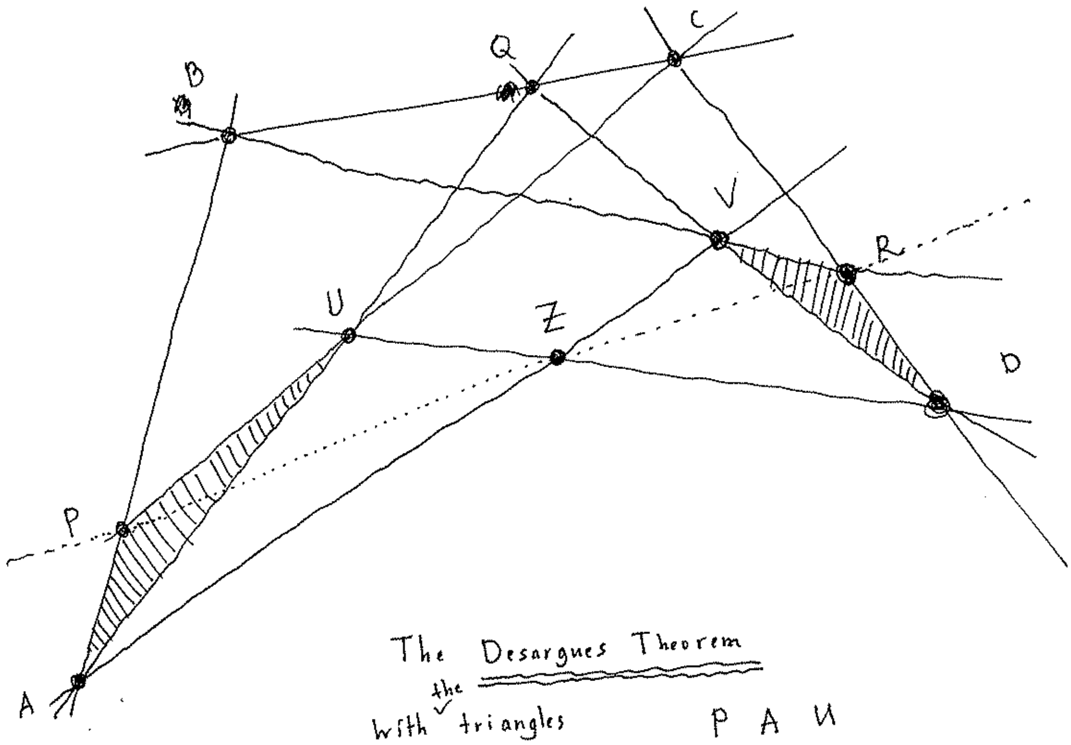
$$a^2 + c^2 - b^2 + 2ac = \left(3 + 2\sqrt{2}\right)\left[b^2 - a^2 - c^2 + 2ac\right]$$

$$\left(4 + 2\sqrt{2}\right)\left(a^2 + c^2 - b^2\right) = \left(2 + 2\sqrt{2}\right) 2ac$$

$$\therefore \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{2 + 2\sqrt{2}}{4 + 2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \underline{B = 45^\circ}$$

2. Given distinct non-collinear points A, B, C, D consider points $P \in AB - \{A, B\}$, $Q \in BC - \{B, C\}$, $R \in CD - \{C, D\}$. Let AQ and PC, BR and QD meet in U, V respectively. Finally, let AV intersect DU in Z . Prove that P, Z, R are collinear.



The Desargues Theorem

with the triangles $\begin{matrix} P & A & U \\ R & V & D \end{matrix}$

Note that $AU \cap VD = \{Q\}$

$UP \cap DR = \{C\}$ & B, C, Q are collinear.

$PA \cap RV = \{B\}$

Hence: RP incides in Z.

(That is UD, AV & PR are concurrent!)

3. In a triangle ABC, let $S \in BC$ be the point in which the incircle touches BC .
(A) Show that $|BS| = s - b$.
(B) Prove that AS and the median through B and the internal angle bisector through C are concurrent iff

$$a^2 + b^2 = (a+b)c$$

(C) Prove that in such a triangle $C \neq \pi/2$.
(D) Such a triangle is equilateral iff it is isosceles.

(A) Routine.

(B) Consider $B' \in CA$, $J \in AB$ such that BB' is the median through B and CJ is the internal angle bisector through C.

AS, BB', CJ are concurrent iff

$$\frac{SB}{SC} \cdot \frac{B'C}{B'A} \cdot \frac{JA}{JB} = \left(-\frac{s-b}{s-c}\right)(-1)\left(-\frac{b}{a}\right) = -1$$

equivalently $\quad b(s-b) = a(s-c)$

$$b(a+c-b) = a(\cancel{a}a+b-c)$$

$$bc - b^2 = a^2 - ac \longleftrightarrow a^2+b^2 = (a+b)c$$

(C) If $C = \pi/2$ then $a^2+b^2 = c^2$ hence $c^2 = a^2+b^2 = (a+b)c$

and $c = a+b$ impossible

unless A, B, C collinear !

(D) If $b = c$ then $a^2+c^2 = a(a+c) = a^2+ac \longrightarrow c^2 = ac$

$\longrightarrow c = a$ ✓

If $c = a$ then $b^2+c^2 = (b+c)c \longrightarrow b^2 = bc \longrightarrow b = c$

If $a = b$ then $2a^2 = 2ac \longrightarrow a = c$.