MATH 373 - GEOMETRY I

SECOND MIDTERM

FAMILY NAME	OTHER NAMES	GRADE

17th December 2010. Duration : 90 minutes. Three questions : 10 + 10 + 20 , 10 + 5 + 10 , 15 + 20

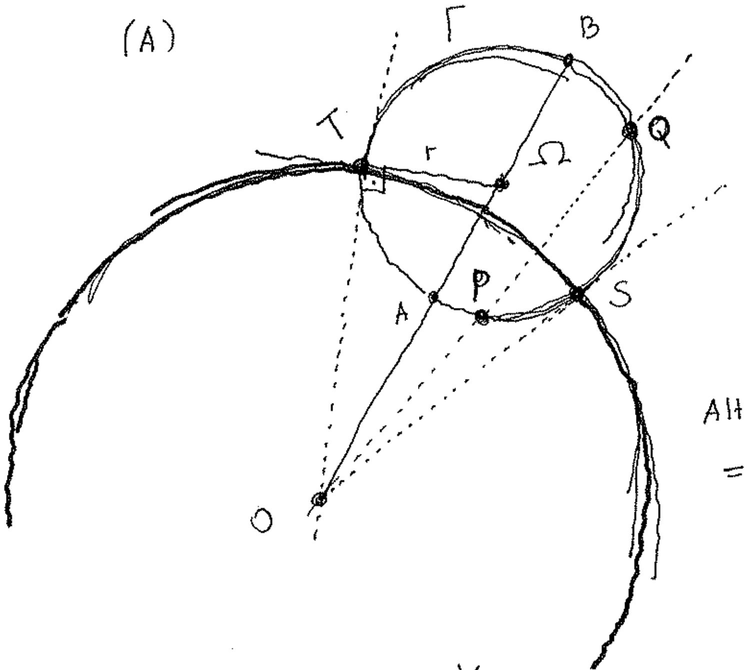
Solutions

1. (A) Let a circle Γ intersect a circle C of center O and radius R orthogonally. Explain briefly why the power of O with respect to Γ is R^2 .

(B) Let a line touch the circles K_1 and K_2 in A, B, respectively. Prove that the radical axis of K_1 and K_2 bisects the line segment $[A, B]$.

(C) Let each one of the circles Γ_1, Γ_2 intersect each one of the circles C_1, C_2 orthogonally. Prove that the radical axis of Γ_1, Γ_2 is the line joining the centres of C_1, C_2 .

(A)



Let Ω and r be the center and the radius of Γ respectively. Let $d = |O\Omega|$.

The power of O w.r.to Γ

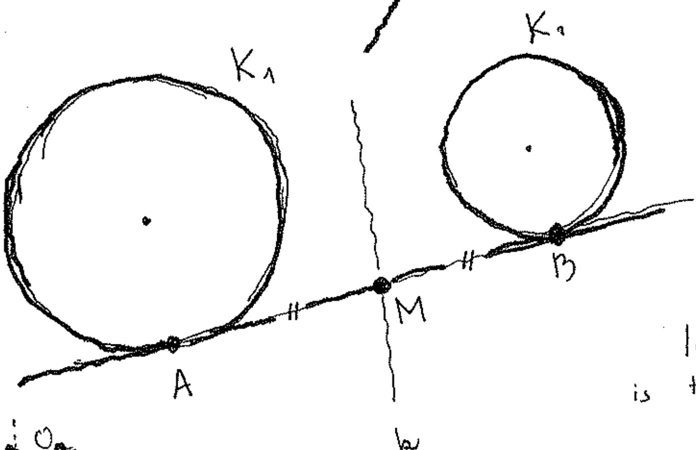
$$= OA \cdot OB = (d+r)(d-r) = d^2 - r^2 = |O\Omega|^2 - |\Omega T|^2 = R^2.$$

Alternatively

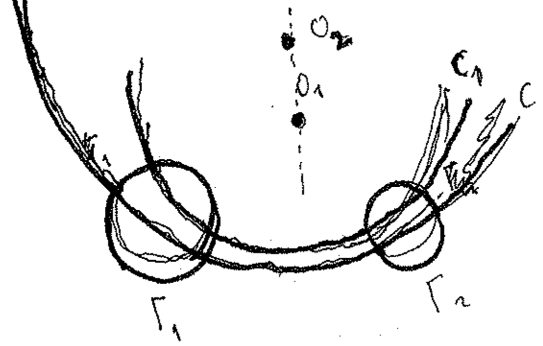
$$= OP \cdot OQ \longrightarrow OS \cdot OS \quad \text{as } P \longrightarrow S.$$

without changing $\quad = R^2$.

(B)



If the radical axis k of K_1 & K_2 intersects AB in M, then the power of M w.r.to $K_1 = |AM|^2$ and w.r.to $K_2 = |BM|^2$ hence $|AM| = |BM|$. As $A \neq B$, M is the midpoint of $[A, B]$



(c) Let R_1, R_2 be the respective radii of C_1, C_2. The power of O_1 w.r.to Γ_1 and Γ_2 is R_1^2. $\therefore$ O_1 is on the radical axis of Γ_1 & Γ_2. Similarly O_2.

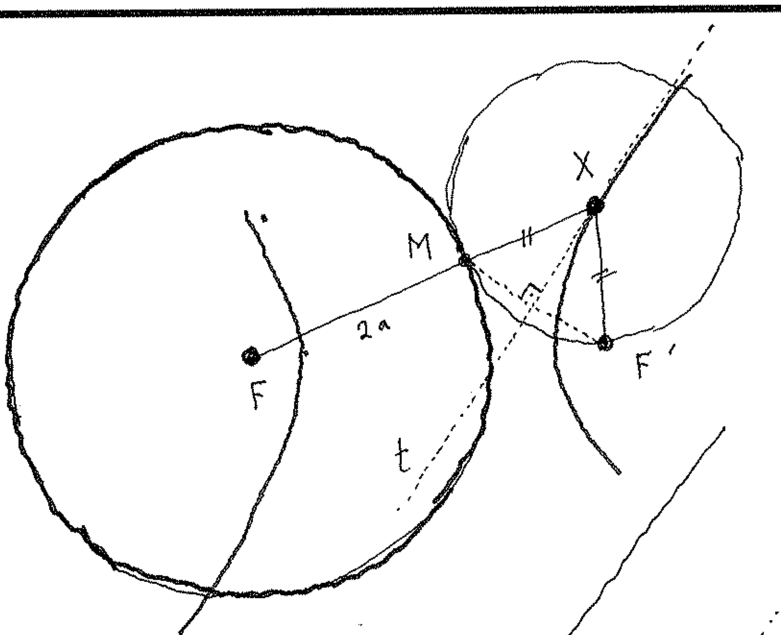
Therefore $O_1 O_2$ is the radical axis of Γ_1 & Γ_2.

2. (A) Let δ be a circle of center F and radius $2a$ and F' be a point with $|FF'| > 2a$. Show that the center X of a circle which touches δ and goes through F' lies on the hyperbola γ of foci F, F' and major diameter $2a$.

(B)Describe (no proof !) the tangent line to γ at X .

(C) With the same notation as (A),(B) above, let k be a line through F' that touches δ in S . Interpret the perpendicular bisector of $[F', S]$ in relation with the hyperbola γ .

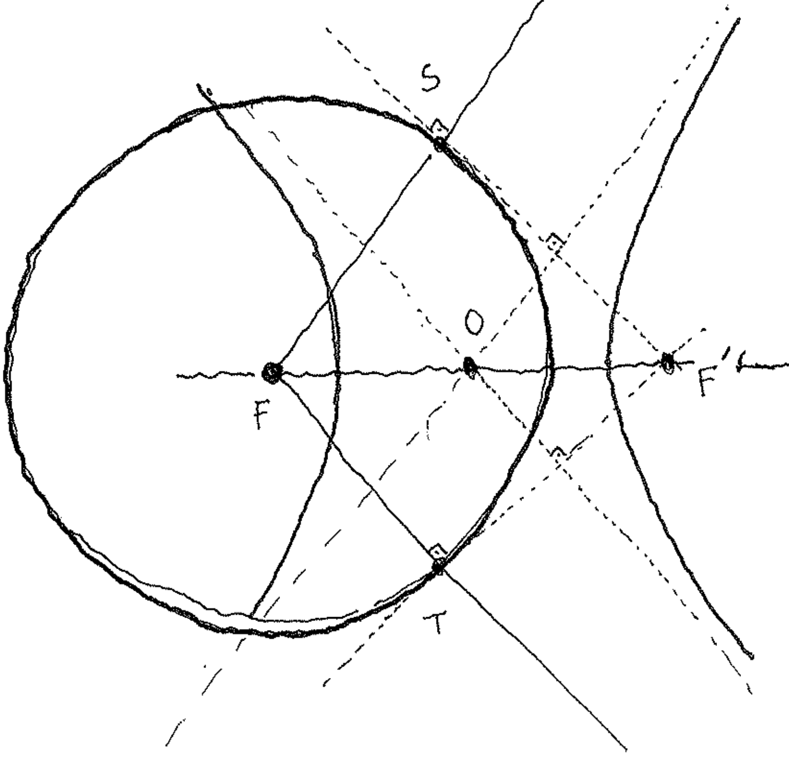
(A, B)



$|XF| - |XF'| = |XF| - |XM|$

$= 2a$

The perpendicular bisector t of $[F'M]$ is the tangent line at X.

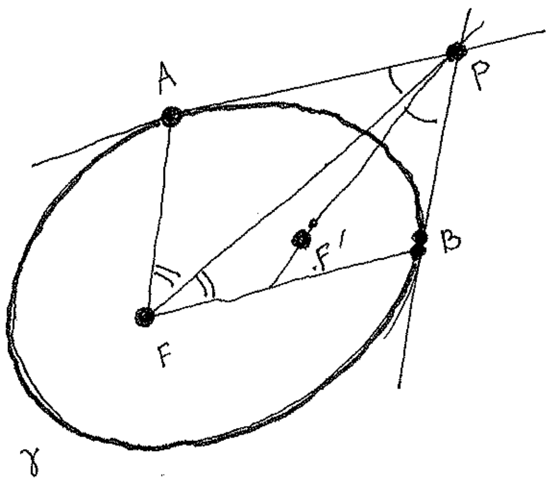


The perpendicular bisectors of $[S, F']$ and $[T, F']$ are the asymptotes of the hyperbola γ.

3. (A) State (do not prove) theorems of Poncelet relating the various angles in a configuration with a conic section, its foci and its tangents. You may assume the conic section to be an ellipse. (Not a circle or a parabola.)

(B) Let φ be an ellipse with foci F, F'. If the tangent lines at $A, B \in \varphi$ intersect in P , prove that $PF \perp FA$ iff $F \in AB$.

(A)

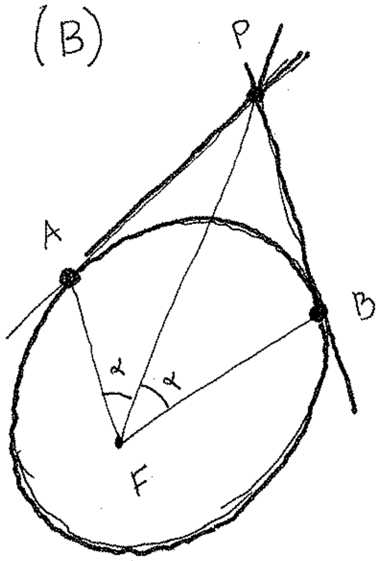


In a conic section φ of foci F, F' if the tangent lines at $A, B \in \varphi$ intersect in P, then

"Poncelet 1" $\measuredangle(PA, PF) = \measuredangle(PF', PB)$

"Poncelet 2" $\measuredangle(FB, FP) = \measuredangle(FP, FA)$.

(B)



Putting $\alpha = \measuredangle(FB, FP) = \measuredangle(FP, FA)$

we note :

1) If $F \in AB$ then

$\measuredangle(FB, FA) = \alpha + \alpha = \pi$. $\therefore$ $\alpha = \pi/2$ i.e $PF \perp FA$

2) If $PF \perp FA$ then $\alpha = \pi/2$

and $\measuredangle(FB, FA) = \alpha + \alpha = \pi$ $\therefore$ $F \in AB$.