Angular Momentum

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) - \frac{d\vec{r}}{dt} \times \vec{p}$$

Since $\vec{p} = m\frac{d\vec{r}}{dt} \parallel \frac{d\vec{r}}{dt}$, $\frac{d\vec{r}}{dt} \times \vec{p} = 0$

Hence

$$\vec{\tau} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}\vec{L}$$

where

$\vec{L} = \vec{r} \times \vec{p}$

is called the angular momentum

Angular Momentum

$\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum of a point object at position $\vec{r}$ and that has momentum $\vec{p}$.

For an extended object

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (m_i \vec{v}_i)$$

$$= \sum_i \vec{r}_i \times (\vec{w} \times \vec{r}_i) m_i$$

$$\vec{L} = \int dv\, \rho(\vec{r})\, \vec{r} \times (\vec{w} \times \vec{r})$$

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$$\vec{L} = \int dV \rho \vec{r} \times (\vec{\omega} \times \vec{r})$$

Using

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{L} = \int dV \rho \left[\vec{\omega} r^2 - \vec{r}(\vec{\omega} \cdot \vec{r})\right]$$

Choose z-axis to be in the $\vec{\omega}$ direction

$$\vec{\omega} = \omega \hat{z}$$

$$\vec{L} = \int dV \rho \omega \hat{z} \left[r^2 - z^2\right] - \int dV \rho z (x\hat{x} + y\hat{y}) \omega$$

$$= I \vec{\omega} - \omega \int dV \rho x z \hat{x} - \int dV \rho y z \omega$$

In general $\vec{L} \neq I \vec{\omega}$

Angular Momentum

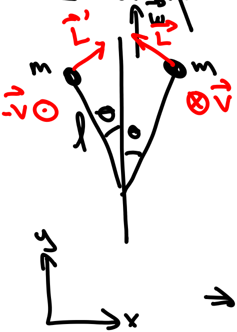
$\vec{\tau} = I\vec{\alpha}$ is valid for objects of sufficient symmetry

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

is valid in general

Angular Momentum

Example



$$|\vec{L}| = |\vec{L}'| = lmv$$

$$\vec{L} = lmv\left[-\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right]$$

$$\vec{L}' = lmv\left[\cos\theta\,\hat{x} + \sin\theta\,\hat{y}\right]$$

$$\vec{L}_T = \vec{L} + \vec{L}' = 2lmv\sin\theta\,\hat{y}$$

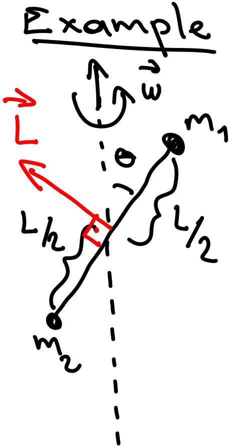
$$|\vec{v}| = wR = wl\sin\theta$$

$$\rightarrow \vec{L}_T = 2lm\sin\theta\; wl\sin\theta\,\hat{y}$$

$$= 2m(l\sin\theta)^2 w\hat{y}$$

$$\vec{L}_T = I\vec{w}$$

Angular Momentum

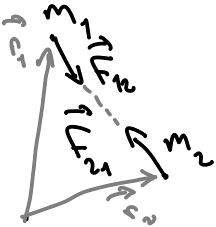


Mass m_1 & m_2 rotate around the z-axis

$$\vec{w} \parallel \hat{z}$$

But angular momentum is perpendicular to the line connecting the two masses.

Conservation of Angular Momentum



Two masses exert forces on each other Then

$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_{12}$ $\quad \vec{\tau}_2 = \vec{r}_2 \times \vec{F}_{21}$

$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12}$

Since $\vec{F}_{12} \parallel (\vec{r}_1 - \vec{r}_2)$

$\vec{\tau} = 0$

The angular momentum of the whole system is conserved

Conservation of Angular Momentum

If there are external forces

$$\vec{\tau}_T^{ext} = \sum_i \vec{r}_i \times \vec{F}_i^{ext}$$

Hence

$$\frac{d\vec{L}}{dt} = \vec{\tau}_T^{ext} \Longleftrightarrow \frac{d\vec{P}_{cM}}{dt} = \vec{F}^{ext}$$

Conservation of Angular Momentum

Previous derivations valid only for fixed axes.

Also valid for angular momentum around CM:

$$\frac{d\vec{L}_{cm}}{dt} = \vec{\tau}_{cm}$$

where $\vec{L}_{cm}$ & $\vec{\tau}_{cm}$ are calculated with respect to the CM.

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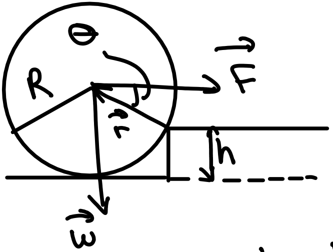
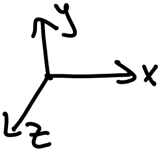
Examples



What is the minimum $\vec{F}$ necessary to roll the sphere up the step?

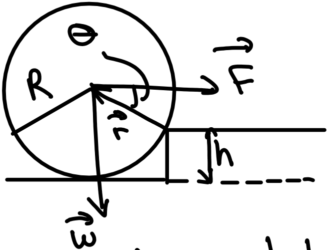
Choose coordinate axis:

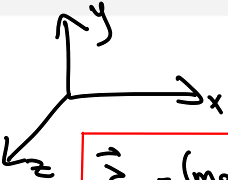


$\vec{F} = F\hat{x}$ $\quad \vec{w} = mg(-\hat{y})$

$\vec{\tau}_F = \vec{r} \times \vec{F} = FR\sin(\pi - \theta)(-\hat{z}) = -FR\sin\theta\,\hat{z}$

$\vec{\tau}_w = \vec{r} \times \vec{w} = Rmg\sin(\frac{\pi}{2} + \theta)\hat{z} = mgR\cos\theta\,\hat{z}$





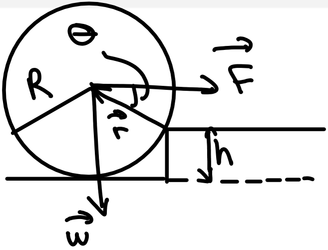
$$\vec{\tau}_F = -FR\sin\theta\,\hat{z}$$

$$\vec{\tau}_w = mgR\cos\theta\,\hat{z}$$

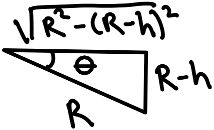
$$\boxed{\vec{\tau}_T = (mgR\cos\theta - FR\sin\theta)\,\hat{z}}$$

If the sphere rotates around the contact point, it should obtain an angular velocity in the $-\hat{z}$ direction. Hence $\vec{\tau}_T$ should create an angular acceleration in the $-\hat{z}$ direction.

$$mgR\cos\theta - FR\sin\theta \leq 0 \Rightarrow F \geq \frac{mg\cos\theta}{\sin\theta}$$



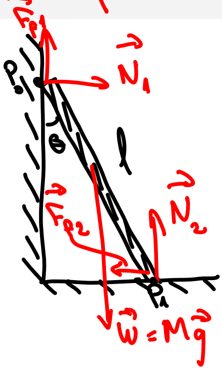
$$F \geq \frac{mg\cos\theta}{\sin\theta}$$



$$F \geq mg\frac{\sqrt{R^2-(R-h)^2}}{R-h}$$

Question: if $h=R$, $F \to \infty$. does this make sense?

Examples: Rod leaning on a wall



Torques with respect to P_0 & P_1 should be zero:

$$P_0: -Mg\,\frac{l}{2}\sin\theta + N_2\,l\sin\theta - F_{f_2}\sin\left(\frac{\pi}{2}+\theta\right) = 0$$

$$P_1: Mg\,\frac{l}{2}\sin\left(\frac{\pi}{2}+\theta\right) - N_1\,l\sin\left(\frac{\pi}{2}+\theta\right) - F_{f_1}\sin(\theta) = 0$$

Net force should be zero:

$$N_1 = F_{f_2}$$

$$N_2 + F_{f_1} = Mg$$

We have four equations for 4 unknowns:

$$F_{f_1},\ F_{f_2},\ N_1,\ N_2$$

they can be solved.