MATH 219

Fall 2020

Lecture 15

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Content: The method of undetermined coefficients.

Suggested Problems: (Boyce, Di Prima, 9th edition)

§4.3: 2, 7, 8, 11, 12, 14, 17, 18, 21

1 Method of undetermined coefficients

Let us now consider a non-homogenous nth order linear equation with constant coefficients, namely an equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = b(t).$$

We can solve any such equation in principle, by using the techniques developed so far: First find all solutions of the corresponding homogenous equation (with $b(t) = 0$), then convert the ODE into a first order $n \times n$ system and apply variation of parameters. This approach will be followed in the next lecture. However, there is a much simpler and direct method if $b(t)$ has a special form. More specifically, let us assume for this lecture that $b(t)$ is a linear combination of functions of type

$$t^k e^{\lambda t}, \qquad t^k e^{at} \cos bt, \qquad t^k e^{at} \sin bt \tag{1}$$

for various values of the nonnegative integer k and real numbers a, b, λ. The strategy can be outlined as follows:

1. Find all solutions $y_h(t)$ of the homogenous equation $y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$.

2. Guess the form of a particular solution $y_p(t)$ of the full equation $y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = b(t)$ as a linear combination of functions in (1). Then find the constants appearing in $y_p(t)$ by placing this function into the ODE.

The description of this method is admittedly vague at this point. Let us work on a few examples first. Afterwards, we will clarify how a correct guess can be made and we will outline a proof of why this method is guaranteed to work.

Example 1.1 *Find all solutions of the ODE*

$$y'' - y = 6e^{3t}.$$

Solution: *The corresponding homogenous equation is* $y'' - y = 0$*. Its characteristic equation is* $\lambda^2 - 1 = 0$*, so* $\lambda = \pm 1$*. Since the roots are distinct, we see that* $y_h = c_1e^t + c_2e^{-t}$*. For a particular solution, it is natural to make a guess of the form* $y_p = Ae^{3t}$*, since the derivatives of* e^{3t} *will again give us multiples of itself and we will have a chance of balancing the left and right hand sides. Plugging* y_p *into the ODE gives*

$$\begin{aligned}(Ae^{3t})'' - Ae^{3t} &= 6e^{3t} \\ 8Ae^{3t} &= 6e^{3t}\end{aligned}$$

and it is clear that $A = \frac{3}{4}$ *works. Therefore*

$$y = y_h + y_p = c_1e^t + c_2e^{-t} + \frac{3}{4}e^{3t}.$$

Example 1.2 *Find all solutions of the ODE*

$$y'' + 3y' + 2y = \cos t.$$

Solution: *Again, we first solve the homogenous equation* $y'' + 3y' + 2y = 0$*. The characteristic equation is* $\lambda^2 + 3\lambda + 2 = 0$ *and its roots are* $\lambda_1 = -2, \lambda_2 = -1$*. Hence* $y_h = c_1e^{-2t} + c_2e^{-t}$*. Let us now think about* y_p*. A guess of the form* $y_p = A\cos t$ *will not be a good idea this time, since the derivatives of this function will also give us* $\sin t$ *terms and it will be impossible to balance them with the term on the right. However, we can set* $y_p = A\cos t + B\sin t$ *and use the extra degree of freedom to balance things out:*

$$\begin{aligned}(A\cos t + B\sin t)'' + 3(A\cos t + B\sin t)' + 2(A\cos t + B\sin t) &= \cos t \\ -A\cos t - B\sin t - 3A\sin t + 3B\cos t + 2A\cos t + 2B\sin t &= \cos t \\ (A + 3B)\cos t + (B - 3A)\sin t &= \cos t.\end{aligned}$$

Since the functions $\cos t, \sin t$ *are linearly independent, such an equation holds for all* t *if and only if the equalities* $A + 3B = 1$ *and* $B - 3A = 0$ *simultaneously hold. This gives us* $A = 1/10$ *and* $B = 3/10$. *So we get*

$$y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10}\cos t + \frac{3}{10}\sin t.$$

Example 1.3 *Find all solutions of the ODE*

$$y'' - y = e^t.$$

Solution: *Just as in the first example of this lecture, we get* $y_h = c_1 e^t + c_2 e^{-t}$. *For the particular solution, a guess of the form* $y_p = Ae^t$ *is not a good idea: Such a term is already present in* y_h. *This means that if we plug it into the left hand side, it will produce* 0 *and can never produce a nonzero term like* e^t. *Therefore we need something else. Inspired by the discussion about repeated roots in previous lectures, we can try* $y_p = Ate^t$. *Then,*

$$\begin{aligned}(Ate^t)'' - Ate^t &= e^t \\ Ate^t + 2Ae^t - Ate^t &= e^t \\ 2Ae^t &= e^t\end{aligned}$$

and we see that $A = \frac{1}{2}$ *works. The important thing here is that the* te^t *terms on the left cancel; if they didn't cancel, it would be impossible to balance the two sides. We obtain*

$$y = y_h + y_p = c_1 e^t + c_2 e^{-t} + \frac{1}{2}te^t.$$

2 Differential operators and annihilators

The examples above give us a glimpse of the method of undetermined coefficients. But at this point it is not absolutely clear how we can make the correct guess and we have no proof that this method will work. In order to close this gap, we will talk about differential operators and annihilators, which are important concepts on their own as well.

Let us develop a convenient notation which is especially useful for constant coefficient linear ODE's. Set

$$D = \frac{d}{dt}.$$

D is called the differentiation operator. We can compose D with itself any number of times and take linear combinations of operators of this type. Such operators are called **differential operators**.

Example 2.1 *By definition, it is clear that* $Dy = y', D^2y = y''$ *etc. Applications of polynomial expressions in* D *to* y *also make sense. For instance:*

$$(D^3 + 4D - 7)y = y''' + 4y' - 7y$$

Based on this new notation, our equation $y^{(n)} + a_1y^{(n-1)} + \ldots + a_ny = b(t)$ can be rewritten as

$$(D^n + a_1D^{n-1} + \ldots + a_n)y = b(t).$$

If we set $p(D) = D^n + a_1D^{n-1} + \ldots + a_n$ then notice that $p(\lambda) = 0$ is the characteristic equation for the corresponding homogenous ODE. Say $\lambda_1, \ldots, \lambda_n$ are the roots of the characteristic equation. Then the differential operator can be factorized and we can write the ODE in the form

$$(D - \lambda_1)(D - \lambda_2)\ldots(D - \lambda_n)y = b(t)$$

2.1 Annihilators

Definition 2.1 *Say* $L(D) = D^k + c_1D^{k-1} + \ldots + c_k$ *is a differential operator. Then* $L(D)$ *is called an* ***annihilator*** *of* $b(t)$ *if*

1. $L(D)b(t) = 0$,
2. $L(D)$ *has the lowest degree among all nonzero polynomial operators such that the first condition is satisfied.*

One can prove that the annihilator of a function is unique if it exists. On the other hand, not all functions have polynomial annihilators. For instance, take $f(t) = 1/t$. Then $Df = -1/t^2$, $D^2f = 2/t^3, \ldots, D^mf = (-1)^m m!/t^{m+1}\ldots$ All these functions are linearly independent, so no nontrivial combination of them is zero. Therefore $f(t) = 1/t$ does not have a polynomial annihilator. Next, we will compute the annihilators of some familiar functions.

Example 2.2 *If* $f(t) = e^{\lambda t}$ *then*

$$(D - \lambda)e^{\lambda t} = \lambda e^{\lambda t} - \lambda e^{\lambda t} = 0.$$

Since $D - \lambda$ *is of first degree, it is necessarily the lowest order operator annihilating* $e^{\lambda t}$*. Therefore the annihilator of* $e^{\lambda t}$ *is* $D - \lambda$*. As a special case, note that the annihilator of* $f(t) = 1$ *is* D*.*

Example 2.3 *Let us take* $f(t) = te^{\lambda t}$*. Then*

$$(D - \lambda)te^{\lambda t} = e^{\lambda t} + \lambda te^{\lambda t} - \lambda te^{\lambda t} = e^{\lambda t},$$

therefore by the previous example

$$(D - \lambda)^2 te^{\lambda t} = (D - \lambda)e^{\lambda t} = 0.$$

The operator $(D - \lambda)^2$ *is of order 2. It can be shown that no first order operator annihilates* $te^{\lambda t}$ *hence the annihilator of* $te^{\lambda t}$ *is* $(D - \lambda)^2$*.*

In a similar manner,

$$(D - \lambda)t^k e^{\lambda t} = kt^{k-1}e^{\lambda t} + \lambda t^k e^{\lambda t} - \lambda t^k e^{\lambda t} = kt^{k-1}e^{\lambda t}.$$

Using this equality and by using induction one can show that the annihilator of $t^k e^{\lambda t}$ *is* $(D - \lambda)^{k+1}$ *for any nonnegative integer* k*.*

Example 2.4 *Let* $f(t) = \cos at$*. Then* $D(\cos at) = -a \sin at$ *and* $D^2(\cos at) = -a^2 \cos at$*. It follows that*

$$(D^2 + a^2)\cos at = 0.$$

It can be shown that no first order differential operator annihilates $\cos at$*, therefore the annihilator of* $\cos at$ *is* $D^2 + a^2$*. A similar computation shows that the annihilator of* $\sin at$ *is also* $D^2 + a^2$*.*

Notice that if $L(D)f(t) = 0$ and $L(D)g(t) = 0$ then $L(D)(c_1 f(t) + c_2 g(t)) = 0$ for any constants c_1, c_2. Furthermore, if $L(D)f(t) = 0$ and $M(D)g(t) = 0$ then $L(D)M(D)(c_1 f(t) + c_2 g(t)) = 0$. Actually, one can be a little bit more economical: If $N(D)$ is the "least common multiple" of $L(D)$ and $M(D)$, then $N(D)(c_1 f(t) + c_2 g(t)) = 0$.

Example 2.5 *Find the annihilator of*

$$f(t) = e^{-2t} + t^2 e^{-2t} + \cos(\sqrt{3}t) + t.$$

Solution: *The annihilators of* $e^{-2t}, t^2 e^{-2t}, \cos(\sqrt{3}t)$ *and* t *are* $D+2, (D+2)^3, D^2+3$ *and* D^2 *respectively. The annihilator of* $f(t)$ *is the least common multiple of all these terms, which is:*

$$(D+2)^3(D^2+3)D^2.$$

Example 2.6 *The annihilator of* $e^{(a+ib)t}$ *is* $D-(a+ib)$ *and the annihilator of* $e^{(a-ib)t}$ *is* $D-(a-ib)$. *Since*

$$e^{at}\cos bt = \frac{e^{(a+ib)t} + e^{(a-ib)t}}{2}$$

we see that $(D-(a+ib))(D-(a-ib)) = D^2 - 2aD + (a^2+b^2)$ *annihilates* $e^{at}\cos bt$. *It can be shown that no first order operator annihilates this function, therefore we found the actual annihilator. Likewise,* $e^{at}\sin bt$ *has the same annihilator.*

Multiplication by t *has the same effect on such functions as for the case of real roots. Therefore a similar analysis shows that the annihilators of* $t^k e^{at}\cos bt$ *and* $t^k e^{at}\sin bt$ *are equal and given by*

$$(D^2 - 2a + (a^2+b^2))^{k+1}.$$

2.2 Method of undetermined coefficients revisited

Let us return to our discussion of a non-homogenous ODE

$$L(D)y = b(t) \tag{2}$$

where $L(D)$ is a linear differential operator with constant coefficients. If $b(t)$ has a polynomial annihilator, then we can solve this equation as follows:

1. Find the annihilator $M(D)$ of $b(t)$.
2. Apply $M(D)$ to both sides of (2) in order to get
$$M(D)L(D)y = 0 \tag{3}$$
3. Find all solutions of (3) (this equation is homogenous, so in principle we know how to do this).

4. Not all solutions of (3) are solutions of (2). To find out which ones are, plug the solution obtained in the previous step back in (2).

The final step will reveal the values of some of the coefficients of the linear combination obtained in step 3. For this reason, the method is named "the method of undetermined coefficients".

Example 2.7 *Solve the ODE*

$$y'' - y = e^{2t}$$

Solution: *The equation can be written as* $(D^2 - 1)y = e^{2t}$ *or*

$$(D-1)(D+1)y = e^{2t}.$$

The annihilator of the right hand side is $D-2$*. Apply this operator to both sides in order to get*

$$(D-2)(D-1)(D+1)y = 0.$$

The three roots of the characteristic equation are $2, 1$ *and* -1*. They are distinct. So, the solutions of this equation are*

$$y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t}.$$

Now let us put this function back in the original equation:

$$\begin{aligned}
(D^2-1)(c_1e^t + c_2e^{-t} + c_3e^{2t}) &= e^{2t}\\
(D^2-1)(c_3e^{2t}) &= e^{2t}\\
4c_3e^{2t} - c_3e^{2t} &= e^{2t}\\
3c_3 &= 1\\
c_3 &= 1/3.
\end{aligned}$$

The fact that $D^2 - 1$ *annihilates* e^t *and* e^{-t} *was used in order to write the second equality. Therefore the solutions of the equation are*

$$y = c_1e^t + c_2e^{-t} + \frac{1}{3}e^{2t}$$

where $c_1, c_2 \in \mathbb{R}$*.*

Example 2.8 *Solve the initial value problem*

$$y''' + 4y' = t, \quad y(0) = y'(0) = 0, y'(0) = 1.$$

Solution: *Rewrite the ODE as*

$$(D^3 + 4D)y = t.$$

The annihilator of the right hand side is D^2. *Apply it to both sides. We get*

$$D^3(D^2 + 4)y = 0.$$

The roots of the characteristic equation are $0, 0, 0, 2i, -2i$. *Therefore all solutions of this new ODE are*

$$y = c_1 \cos 2t + c_2 \sin 2t + c_3 + c_4 t + c_5 t^2.$$

Plug this function back in the original ODE:

$$\begin{aligned} (D^3 + 4D)(c_1 \cos 2t + c_2 \sin 2t + c_3 + c_4 t + c_5 t^2) &= t \\ (D^3 + 4D)(c_4 t + c_5 t^2) &= t \\ 4c_4 + 8c_5 t &= t. \end{aligned}$$

From the last equality we deduce that $c_4 = 0, c_5 = 1/8$. *Therefore all solutions of the ODE are*

$$y = c_1 \cos 2t + c_2 \sin 2t + c_3 + \frac{1}{8}t^2.$$

We still need to use the initial values in order to find the remaining constants. Compute $y' = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{4}t$ *and* $y'' = -4c_1 \cos 2t - 4c_2 \sin 2t + \frac{1}{4}$. *The initial conditions imply*

$$\begin{aligned} c_1 + c_3 &= 0 \\ 2c_2 &= 0 \\ -4c_1 + \frac{1}{4} &= 1. \end{aligned}$$

We get $c_1 = -3/16$, $c_2 = 0$ *and* $c_3 = 3/16$ *as the unique solution of this linear system. Therefore*

$$y = -\frac{3}{16}\cos 2t + \frac{3}{16} + \frac{1}{8}t^2.$$

Example 2.9 *Find the form of the solutions of*

$$y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t.$$

In other words, write the solution as a linear combination of several terms and point out which coefficients will be determined (do not compute them).

Solution: *Rewrite the equation as*

$$(D^2 + 2D + 2)y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t.$$

The annihilators of $e^{-t}, e^{-t}\cos t$ *and* $e^{-t}t^2 \sin t$ *are* $D + 1, D^2 + 2D + 2$ *and* $(D^2 + 2D + 2)^3$ *respectively. Therefore the annihilator of the right hand side is their least common multiple, which is*

$$(D + 1)(D^2 + 2D + 2)^3.$$

Apply this operator to both sides in order to get

$$(D + 1)(D^2 + 2D + 2)^4 y = 0.$$

The solutions of this 9th order equation are

$$\begin{aligned} y = {} & c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + c_3 t e^{-t}\cos t + c_4 t e^{-t}\sin t + c_5 t^2 e^{-t}\cos t \\ & + c_6 t^2 e^{-t}\sin t + c_7 t^3 e^{-t}\cos t + c_8 t^3 e^{-t}\sin t + c_9 e^{-t}. \end{aligned}$$

When we plug this back in the original equation, the first two summands will be annihilated by $D^2 + 2D + 2$*, hence* c_1 *and* c_2 *will be free. The other 7 constants* $c_3, c_4, \ldots, c_9$ *are to be determined.*