MATH 219

Fall 2020

Lecture 17

Lecture notes by Özgür Kişisel

Content: Mechanical Systems

Suggested Problems: (Boyce, Di Prima, 9th edition)

§3.7: 3,7,12,18,26,27

§3.8: 4,6,10,11,15,17

In this lecture, we will study some important applications of our results about higher order linear equations. Many real life problems concern vibrations of a mechanical or electrical system. Differential equations provide an excellent setting for understanding them.

1 Undamped Spring-Mass Systems

Suppose that we have a linear spring. Hooke's law asserts that the restoring force applied by the spring is directly proportional to the contraction/elongation distance of the spring. We assume that one end of the spring is firmly attached to a ceiling. If we attach an object of mass m to the other end, then the spring will be elongated, say by ℓ meters. If the object is not moving (in other words, if it is at equilibrium), then the gravitational force on the object must be equal to the restoring force exerted by the spring. Therefore,

$$mg = k\ell.$$

This relation can be used to determine the spring constant k, if it is unknown. Suppose now that the object is not at equilibrium, so it is moving up and down as the spring contracts and elongates. We would like to understand precisely how the object moves. Suppose that $u(t)$ denotes its position at time t. We may assume that the equilibrium position corresponds to $u = 0$.

When the object is at position $u(t)$, the force exerted by the spring on it (apart from the $k\ell$ term, which is equalized by mg) is $ku(t)$. Therefore, by Newton's law

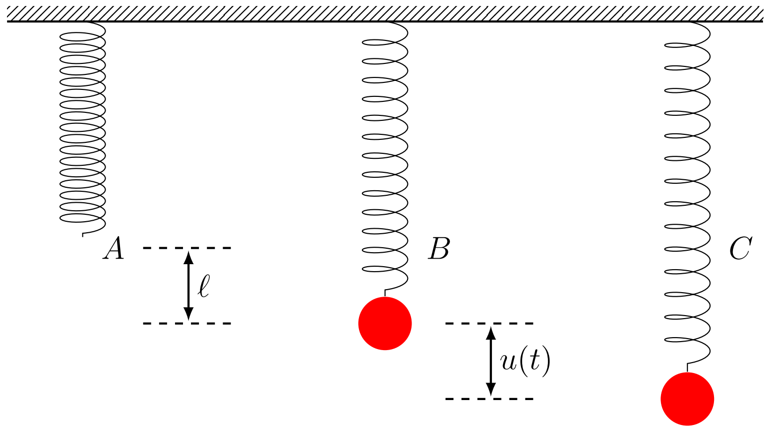


Figure 1: Diagram A: Nothing is attached to the spring. Diagram B: An object of mass m is attached to the spring. It causes an elongation of ℓ units, however the object is not moving. Diagram C: The object is moving. The displacement from the position in the previous diagram is $u(t)$.

of motion $F = ma$, we may write

$$mu'' = -ku.$$

Let us put all terms of this equation on the left hand side and write

$$mu'' + ku = 0.$$

This is a constant coefficient, linear, homogenous ODE for u. The characteristic equation is $m\lambda^2 + k = 0$ and its roots are $\lambda_{1,2} = \pm i\sqrt{k/m}$. It is customary to write $\omega_0 = \sqrt{k/m}$ and to call it the **natural frequency** of the system. The general solution of the equation is:

$$u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

where $c_1, c_2 \in \mathbb{R}$.

The values of c_1 and c_2 are determined by the initial conditions $u(t_0)$ and $u'(t_0)$. There is another way to write the same solution which is handy for graphing it. Write the point (c_1, c_2) in $\mathbb{R}^2$ in polar coordinates. Namely, let

$$R = \sqrt{c_1^2 + c_2^2}, \quad c_1 = R\cos\theta, \quad c_2 = R\sin\theta.$$

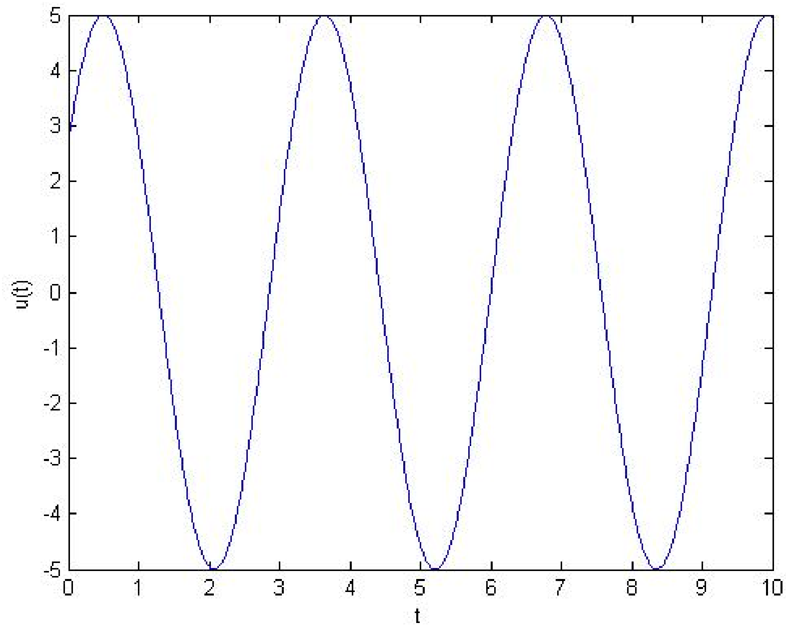
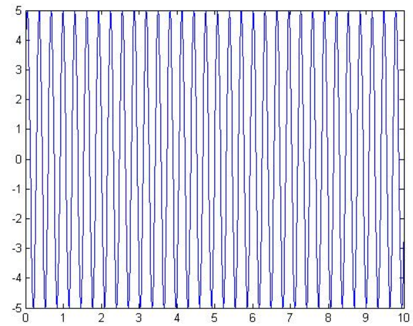


Figure 2: Output of an undamped system. Here, $u(t) = 5cos(2t - 1)$.

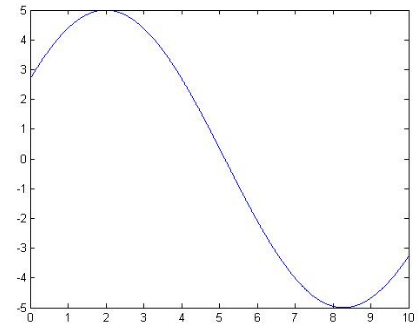
Then we can write $u(t)$ as follows:

$$\begin{aligned} u(t) &= c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ &= R(\cos\theta \cos(\omega_0 t) + \sin\theta \sin(\omega_0 t)) \\ &= R\cos(\omega_0 t - \theta). \end{aligned}$$

This equation shows that the maximum and minimum values attained by $u(t)$ are R and $-R$ respectively. The magnitude R is called the **amplitude** of the wave. The graph is a cosine function shifted according to the value of θ. The angle θ is called the **phase**. One can see the effect of the natural frequency ω_0 on the graph: A high frequency (large ω_0) gives us a denser graph and a low frequency (small ω_0) gives us a sparser graph.



(a) High frequency ($\omega_0 = 20$)



(b) Low frequency ($\omega_0 = 0.5$)

2 Effect of Damping

Suppose that, in addition to the forces above, a damping force (which could be due to friction or some viscous medium, etc.) applies on the object. We will assume that this damping force is directly proportional to the velocity of the object. Suppose that the proportionality constant is γ. Then,

$$mu'' = -ku - \gamma u'.$$

This equation is again a linear, homogenous equation with constant coefficients. If we move all of its terms to left hand side, then it becomes $mu'' + \gamma u' + ku = 0$. Notice that all of the coefficients m, γ, k are positive. The characteristic equation and its roots are:

$$m\lambda^2 + \gamma\lambda + k = 0$$
$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

The form of the solution depends on the signature of the discriminant $\Delta = \gamma^2 - 4mk$.

Overdamped case ($\Delta > 0$): In this case both eigenvalues λ_1, λ_2 are real and $\lambda_1 \neq \lambda_2$. We claim that both eigenvalues are negative. Indeed, notice that

$$\lambda_1 + \lambda_2 = -\frac{\gamma}{m} < 0, \quad \lambda_1\lambda_2 = \frac{k}{m} > 0.$$

Since two real numbers whose sum is negative and whose product is positive must both be negative, we see that both eigenvalues are negative.

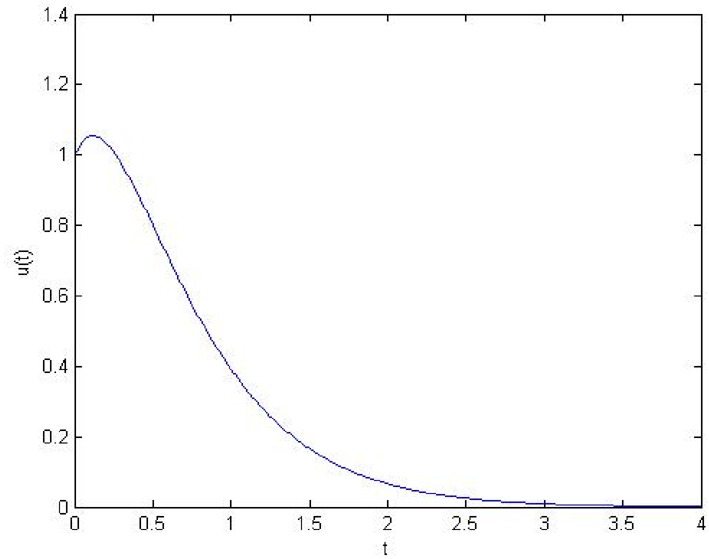


Figure 4: Output of an overdamped system. Here, $u(t) = 4e^{-t} - 3e^{-2t}$.

The solution $u(t)$ of the equation is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

where c_1, c_2 are arbitrary constants. Since λ_1, λ_2 are negative, $u(t)$ tends to 0 as t tends to $+\infty$. It is not difficult to see that the function $u(t)$ has at most one local maximum or minimum.

Critically damped case ($\boldsymbol{\Delta = 0}$): In this case, $\lambda_1 = \lambda_2 = -\gamma/2m$. Notice that this number is negative. The solution is

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}$$

where c_1, c_2 are arbitrary constants. Again, it is easy to check that $\lim_{t \to +\infty} u(t) = 0$ and the function $u(t)$ has at most one local maximum or minimum.

Underdamped case ($\boldsymbol{\Delta < 0}$): In this case λ_1 and λ_2 are a pair of complex conjugate numbers given by:

$$\lambda_{1,2} = -\frac{\gamma}{2m} \pm i\frac{\sqrt{-\Delta}}{2m}.$$

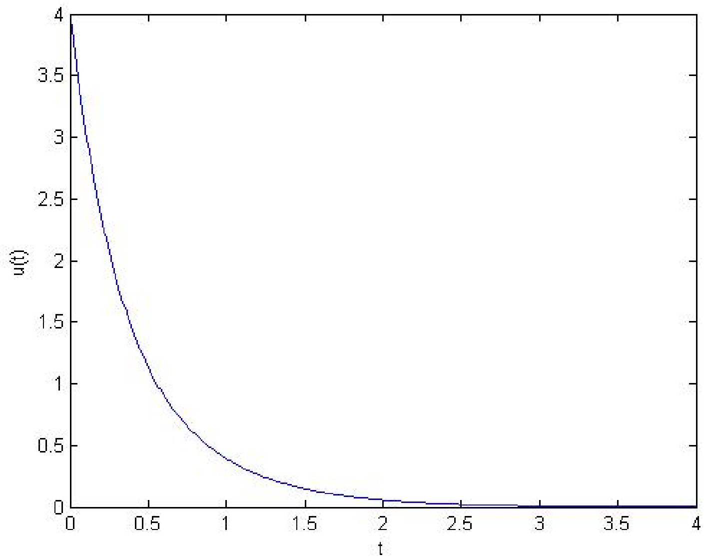


Figure 5: Output of a critically damped system. Here, $u(t) = 4e^{-2t} - 3te^{-2t}$.

Let us say $\omega = \sqrt{-\Delta}/2m$ for convenience. The general solution of the system is

$$u(t) = c_1 e^{-\frac{\gamma}{2m}t}\cos(\omega t) + c_2 e^{-\frac{\gamma}{2m}t}\sin(\omega t)$$

where c_1, c_2 are arbitrary real numbers. As in the undamped case, if we express (c_1, c_2) in poılar coordinates, we get

$$u(t) = Re^{-\frac{\gamma}{2m}t}\cos(\omega t - \theta).$$

Here, $c_1 = R\cos\theta$ and $c_2 = R\sin\theta$ as before. The curve will be trapped between the "envelope curves" $y = Re^{-\frac{\gamma}{2m}t}$ and $y = -Re^{-\frac{\gamma}{2m}t}$, and it will oscillate between them. The function $u(t)$ has infinitely many local maxima and minima, which corresponds to a never-ending oscillatory motion whose amplitude drops towards 0 in time.

3 Mechanical Systems with a Forcing Term

Suppose now that, in addition to the forces previously considered, an external force $F(t)$ acts on the object. It is important for the discussion that this force is not

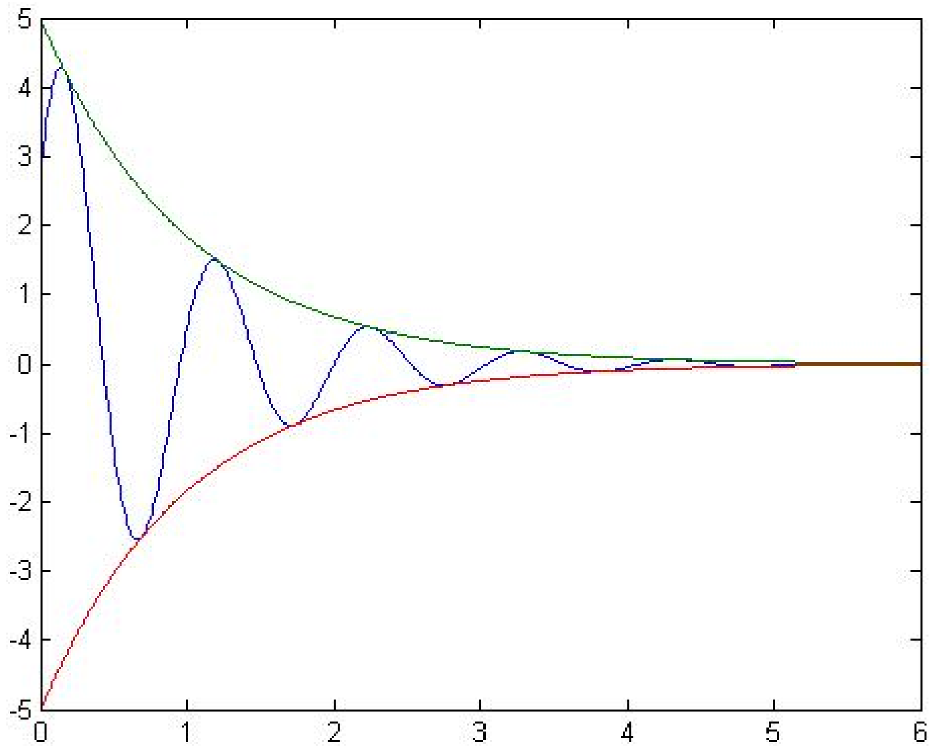


Figure 6: Output of an underdamped system. Here $u(t) = 5e^{-t}\cos(6t - 1)$. The envelope curves are also drawn.

explicitly dependent on the position of the object, but it is dependent on time. The equation of motion becomes:

$$mu'' + \gamma u' + ku = F(t).$$

Therefore, the equation is still linear with constant coefficients, but this time it is non-homogenous. Let $u_h = c_1u_1 + c_2u_2$ be the solution of the corresponding homogenous equation. It can be found as discussed in the previous sections of this lecture. Having found $u_h(t)$, we can apply the variation of parameters formula in order to find $u(t)$. Let $W = u_1u_2' - u_1'u_2$ be the Wronskian of u_1 and u_2. Then,

$$u(t) = u_1(t)\int \frac{-u_2(t)F(t)/m}{W}dt + u_2(t)\int \frac{u_1(t)F(t)/m}{W}dt.$$

Notice that we need to write $F(t)/m$ and not $F(t)$ in the formula since the leading coefficient of the ODE is m and not 1.

This formula is quite significant: It expresses the "output" $u(t)$ in terms of the "input" $F(t)$ in a condensed way. It is very general, essentially it tells us how any such system responds to an input, and this is given as a formula in terms of the system parameters. Now, let us investigate a special case in much more detail.

3.1 A special case

Let us assume for the sake of this discussion that $\gamma = 0$ (the system is undamped) and $F(t) = \cos(\omega t)$. Furthermore assume that the external frequency ω is not equal to the natural frequency $\omega_0 = \sqrt{k/m}$. The ODE in question is:

$$mu'' + ku = \cos(\omega t).$$

We know that the homogenous part of the solution is

$$u_h(t) = c_1 u_1(t) + c_2 u_2(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

Instead of employing the variation of parameters formula, let us solve this equation by the method of undetermined coefficients. Set $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$. Plugging $u_p(t)$ into the ODE gives

$$m(-\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t)) = \cos(\omega t).$$

Equating the coefficients of $\cos(\omega t)$ on both sides and the coefficients of $\sin(\omega t)$ on both sides, we get:

$$A = \frac{1}{k - m\omega^2} = \frac{1}{m(\omega_0^2 - \omega^2)}, \qquad B = 0.$$

Therefore,

$$u(t) = u_h(t) + u_p(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{1}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Notice that the amplitude of the particular solution $u_p(t)$ is highly sensitive to the difference between the external frequency ω and the natural frequency ω_0. If we draw the graph of the amplitude $|A|$ with respect to ω, we see this clearly. The phenomenon of the amplitude going to ∞ when ω tends to ω_0 is called **resonance**. Notice that the input amplitude is always 1 in this example, however the output amplitude changes dramatically with ω. This fact has many applications and also poses many potential dangers in real life situations.

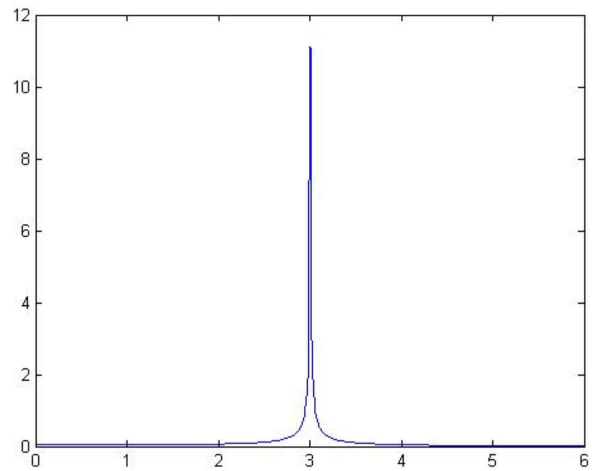


Figure 7: This figure shows the frequency response of a system where $\omega_0 = 3$. The horizontal axis is the external frequency ω and the vertical axis is the amplitude of the output.

4 Further Details

Below, we collect some remarks which can be ignored in the first reading but would be interesting after the material in the previous sections are digested by the reader.

1. Let us look at the special case that we discussed above in some more detail. Suppose that the system is initially "at rest", namely $u(0) = u'(0) = 0$. Then, first of all, $u'(0) = 0$ implies that $c_2 = 0$. Then, using $u(0) = 0$ we get $c_1 = -1/m(\omega_0^2 - \omega^2)$. Therefore,

$$\begin{aligned} u(t) &= \frac{1}{m(\omega_0^2 - \omega^2)}(\cos(\omega t) - \cos(\omega_0 t)) \\ &= \frac{-2}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 + \omega)t}{2}\right) \sin\left(\frac{(\omega - \omega_0)t}{2}\right). \end{aligned}$$

   For the last equality, use the trigonometric identity relating the difference of two cosines to a product of sines. If ω is close to ω_0, but not equal to it, then this function is the product of a low frequency wave and a high frequency wave. The low frequency graph will act as an envelope function for the high frequency oscillations. The low frequency envelope waves are called **beats**.

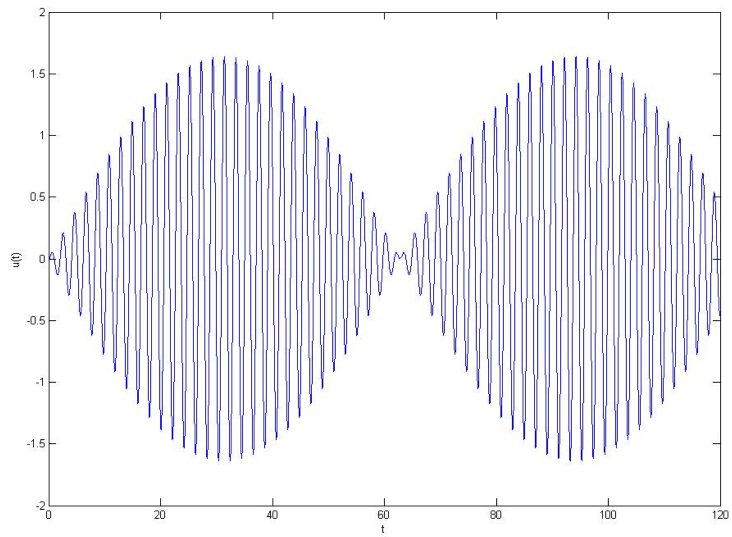


Figure 8: This is the output of an undamped system with $\omega_0 = 3$ and the external frequency $\omega = 3.1$. Beats can be observed clearly in this graph.

2. If $\omega = \omega_0$ in the discussion above, then we should try a particular solution in a different form: $u_p(t) = At\cos(\omega_0 t) + Bt\sin(\omega_0 t)$. The graph will again be similar to the beats graph described above, but this time the envelope will be a linear function rather than a sinusoidal. The amplitude of $u(t)$ is infinite.

3. From a "black-box" point of view, the system takes the function $F(t)$ as input and produces $u(t)$ as output. The relation between $u(t)$ and $F(t)$ can be written using the variation of parameters formula. Linearity of the ODE implies the following: Suppose that the input $F(t)$ produces an output $u(t)$ and the input $\hat{F}(t)$ produces an output $\hat{u}(t)$. Then the input $c_1F(t) + c_2\hat{F}(t)$ will produce the output $c_1u(t) + c_2\hat{u}(t)$.

   The fact that the ODE has constant coefficients m, γ, k can be interpreted as the system parameters not changing in time. We say that the system is "time invariant".

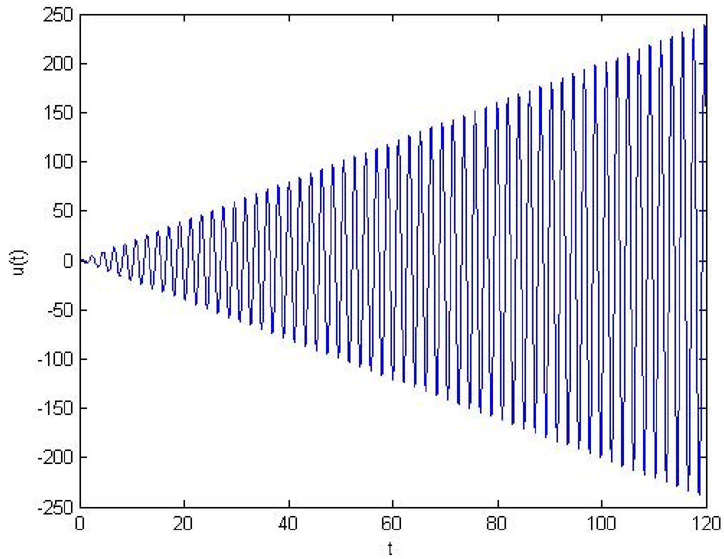


Figure 9: Here, $u(t) = 2t\cos(3t-1)$. Resonance occurs.

4. In the discussion of undamped systems, the case of $\Delta = 0$ was called the "critically damped case", but we did not say much about why it is called critical. This is related to how fast the solution $u_h(t)$ decays to 0 as $t \to +\infty$. Here are some basic rules of thumb:

   - If r_1, r_2 are both positive real numbers, then $e^{-r_1 t}$ decays to 0 faster than $e^{-r_2 t}$ if $r_1 > r_2$.
   - If $u_h(t)$ is the sum of several terms, then the decay rate of the **slowest** decaying summand determines the rate of decay of $u_h(t)$.

   Going back to $mu'' + \gamma u' + ku = 0$, suppose that m and k are kept constant but γ is changing. Suppose first that the roots of the characteristic equation are real, namely the system is either overdamped or critically damped. Since

   $$\lambda_1 + \lambda_2 = -\frac{\gamma}{m},$$

   one of them is at least as large as their average, $-\gamma/2m$. The larger root will determine the rate of decay. Therefore the fastest decay among all these possibilities is the case of critical damping, $\lambda_1 = \lambda_2 = -\gamma/2m$.

It remains to look at the underdamped case. In this case, $|\lambda_1| = |\lambda_2|$ and their product $\lambda_1\lambda_2 = |\lambda_1|^2 = k/m$ does not change with γ. Hence $|\lambda_1| = |\lambda_2| = \sqrt{k/m}$. The real parts of λ_i will be smallest when the roots are real, namely at the point of critical damping. Since the real parts determine the rate of decay, we again see that the underdamped cases decay slower compared to the critically damped case.

Summarizing, for fixed values of m and k, the critically damped case is the one where u decays to zero fastest.

5. Let us look at a damped system with an external force $F(t)$. If the force $F(t)$ is not tending to 0 when $t \to \infty$, then neither will $u_p(t)$. However, we know that $\lim_{t\to\infty} u_h(t) = 0$. Therefore, for practical reasons, only $u_p(t)$ will remain after a while and we won't see much of $u_h(t)$. For this reason, $u_h(t)$ is called the **transient** part of the solution and $u_p(t)$ a **steady state** solution. Notice that this is not entirely well-defined, since shifting part of u_h into the particular solution creates an equally valid particular solution.

6. The whole discussion has a very similar counterpart in the theory of electrical circuits. The relevant electrical components for this discussion are **linear components**. These components and their voltage-current relationships are

   - Resistors ($V = IR$)
   - Capacitors ($I = C\frac{dV}{dt}$)
   - Inductors ($V = L\frac{dI}{dt}$)

   In these equations, V denotes the voltage drop across the relevant component and I denotes the current through it. The values R, C, L are assumed to be constants. Some other circuit components such as diodes, transistors or op-amps are not linear components and cannot be analyzed by using linear differential equations.

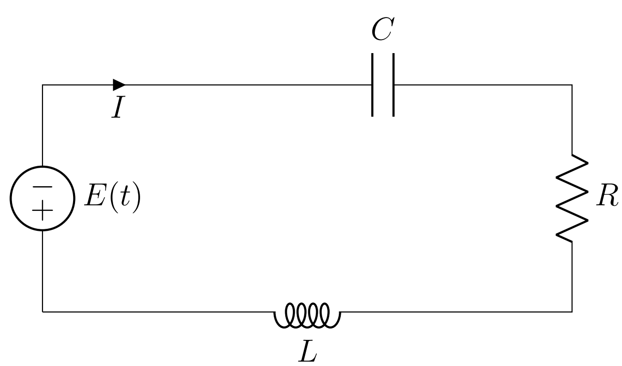
   The correct counterpart of the external force $F(t)$ is a voltage or current source. Putting these components into an electrical circuit gives us differential equations relating the currents and voltages which one can afterwards solve. For instance, suppose that we have a series diagram containing one component of each type and a voltage source as in the figure below. Then, $V_C + V_L + V_R = E(t)$ since both sides measure the potential difference between two points in two different ways. Differentiating this relation once and using the voltage-current relations for each component, we get

$$\begin{aligned}\frac{dV_C}{dt} + \frac{dV_L}{dt} + \frac{dV_R}{dt} &= \frac{dE}{dt}\\ \frac{I}{C} + L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} &= \frac{dE}{dt}.\end{aligned}$$

This is a second order, linear, constant coefficient, non-homogenous ODE for the current $I(t)$ through the circuit. Therefore, the theory should be exactly parallel to the case of a spring mass system with an external forcing term. The role of $F(t)$ is played by $E'(t)$. The resistor causes damping, hence $R = 0$ corresponds to the undamped case. In the undamped case, the natural frequency will be

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

For instance, if R is very low and $E(t)$ contains a summand with frequency very close to ω_0, then one should expect a current with very large amplitude.



7. Even though we discussed second order ODE's in this lecture predominantly, everything can be generalized to higher orders easily. Writing down the equations as a system rather than a single equation may occasionally be more convenient. Higher order ODE's could come from spring-mass systems with more springs and components interconnected to each other. Likewise, higher order ODE's could arise from more complicated electrical circuits than the one we discussed above. We should note that in a realistic application the number of such components could very well be at the order of thousands or more. Therefore, numerical techniques for operating with large matrices become important.

8. Oscillations occur everwhere in real life. For this reason, the general principles outlined in this lecture tend to reccur in many different settings. The reader is encouraged to think about the validity and/or reason of the following statements and make up some of his/her own:

   - Resonance can cause damage in large constructions. There are some examples such as the Tacoma bridge and the Millenium bridge incidents.
   - Potential damage of resonance on constructions can be avoided by arranging the damping parameters at the stage of design.
   - In order to lessen damage during an earthquake it is important to understand the possible frequencies supplied by the earthquake.
   - When you take your car to a mechanic, the suspension system should be tuned so that it is at critical damping.
   - It may be possible to crack a kidney stone externally, without any surgery or damage to the surrounding tissues.
   - A glass can be broken with an appropriate sound wave.
   - A piano tuner listens to the difference between two close but unequal notes via the resulting beat.
   - In old style radios, turning the frequency knob changes a capacitance C, therefore the natural frequency of a circuit.
   - If we wish to filter out certain frequencies of an incoming signal, we can do so by building a suitable electrical circuit.