MATH 219

Fall 2020

Lecture 24

Lecture notes by Özgür Kişisel

Content: Impulse Function and Convolution.

Suggested Problems (Boyce, Di Prima, 9th edition):

§6.5: 3, 7, 10, 15, 19

§6.6: 2, 4, 10, 11, 17, 20, 21, 22

1 Impulse Function

Let us start with a question: Does there exist a function $\delta(t)$ whose Laplace transform is the constant function 1? So far, in all examples that we have seen, Laplace transforms tend to 0 when s becomes large. Therefore if such a function exists, then it must be something that didn't appear before.

One motivation for the answer comes from the derivative formula $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$. If we take $f(t) = u_0(t)$, the unit step function at 0, then its Laplace transform is $1/s$, therefore

$$\mathcal{L}\{u_0'(t)\} = s \cdot \frac{1}{s} - u_0(0) = 1 - u_0(0).$$

We defined $u_0(0)$ to be equal to 1, but we could also have defined it to be 0 at this discontinuity, and this would apparently give us a Laplace transform equal to 1 above. The more serious problem in this argument, of course, is that $u_0(t)$ is not a differentiable function. The clever idea, essentially due to Heaviside, was to define a "fictitious" derivative $\delta(t)$ of $u_0(t)$ and to use this generalized function. Much later, the idea was completely formalized by Schwartz, by developing the theory of distributions.

More rigorously, let us consider a sequence of functions $f_n(t)$ defined as follows:

$$f_n(t) = \begin{cases} n, & \text{if } 0 \le t < \frac{1}{n}, \\ 0, & \text{otherwise} \end{cases}$$

Notice that as n increases, the maximum height of the function $f_n(t)$ increases, but the width of the interval where is attains this height decreases, and the area under the curve is always 1. Expressing $f_n(t)$ in terms of unit step functions, we get

$$f_n(t) = n(u_0(t) - u_{1/n}(t)).$$

Therefore, we can calculate its Laplace transform to be:

$$\mathcal{L}\{f_n\} = \frac{n(1 - e^{-s/n})}{s}.$$

Even though the functions $f_n(t)$ do not have an honest limit function for $n \to \infty$, their Laplace transforms do have a nice limit. Indeed,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{L}\{f_n\} &= \lim_{n\to\infty} \frac{n(1 - e^{-s/n})}{s} \\
&= \lim_{x\to 0} \frac{1 - e^{-sx}}{xs} \\
&= \lim_{x\to 0} \frac{se^{-sx}}{s} \\
&= 1
\end{aligned}$$

In this calculation, one first makes a change of variables $x = 1/n$ and then l'Hospital's rule is used to resolve the 0/0 indeterminacy. Returning to our original question, the role of a function $\delta(t)$ whose Laplace transform is 1 is played by the limit of $f_n(t)$'s as $n \to \infty$, which does not exist in the set of functions. The solution (at least naively) is to add this limit object $\delta(t)$ formally to the set of functions, and get the impulse function $\delta(t)$ as a "generalized function". Schwartz's work formalizes this process and shows that one obtains a consistent theory of generalized functions.

Shifting this function $\delta(t)$ by c units, we obtain the function $\delta(t - c)$, which is an impulse function taking place at time c. Such a function can be used to model an instantaneous external force supplied to the system at time c.

In terms of solving equations with an impulsive forcing term, there is nothing new. Just take the Laplace transforms of both sides of the equation and proceed as before.

Example 1.1 *Solve the initial value problem*

$$y'' + 9y = 15\delta(t - 3\pi) + 12\delta(t - 6\pi), \quad y(0) = y'(0) = 0.$$

Also, find $y(13\pi/2)$.

Solution: *Take Laplace transforms of both sides of the equation.*

$$\begin{aligned} \mathcal{L}\{y''\} + 9\mathcal{L}\{y\} &= 15\mathcal{L}\{\delta(t-3\pi)\} + 12\mathcal{L}\{\delta(t-6\pi)\} \\ (s^2+9)\mathcal{L}\{y\} &= 15e^{-3\pi s} + 12e^{-6\pi s} \\ \mathcal{L}\{y\} &= \frac{15e^{-3\pi s}}{s^2+9} + \frac{12e^{-6\pi s}}{s^2+9} \end{aligned}$$

Now, take inverse Laplace transforms of both sides. By using the theorem about time shifts discussed in lecture 22, we have

$$y(t) = 5u_{3\pi}(t)\sin(t-3\pi) + 4u_{6\pi}(t)\sin(t-6\pi).$$

Finally,

$$y(13\pi/2) = 5\sin(7\pi/2) + 4\sin(\pi/2) = -1.$$

2 Convolution

Although Laplace transform is linear, it is not multiplicative. In other words, $\mathcal{L}\{f\cdot g\}$ is in general not equal to the product of the Laplace transforms $\mathcal{L}\{f\}$ and $\mathcal{L}\{g\}$. This raises a natural question: What is the operation in one domain that corresponds to multiplication in the other? The answer is given by **convolution** which we define below:

Definition 2.1 *Suppose that $f(t)$ and $g(t)$ are two piecewise continuous functions. Their **convolution**, denoted by $f \star g$, is the function given by the formula*

$$(f \star g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

In order to understand this definition in more detail, let us compute an example:

Example 2.1 *Compute the convolution of $f(t) = u_2(t)$ and $g(t) = u_5(t)$.*

Solution:

$$(f \star g)(t) = \int_0^t u_2(t-\tau)u_5(\tau)d\tau.$$

Notice that $u_2(t-\tau)$ is nonzero iff $t-\tau \geq 2$ and $u_5(\tau)$ is nonzero iff $5 \leq \tau$. In particular, if $t < 7$ then these two intervals are disjoint, hence the product of these two functions is identically 0. Therefore the value of the integral is 0 as well.

Suppose now that $t \geq 7$. Then

$$(f \star g)(t) = \int_0^t u_2(t-\tau)u_5(\tau)d\tau = \int_5^{t-2} 1 d\tau = t-7.$$

Therefore we deduce that

$$(f \star g)(t) = \begin{cases} 0, & t < 7 \\ t-7, & t \geq 7 \end{cases}$$

One can also express $f \star g$ in terms of unit step functions:

$$(f \star g)(t) = (t-7) \cdot u_7(t).$$

As advertised before, the key property of convolution is its relation to the Laplace transform:

Theorem 2.1 *(Convolution Theorem) Suppose that $f(t)$ and $g(t)$ are piecewise continuous functions whose Laplace transforms converge for $Re(s) > a$. Then the Laplace transform of $f \star g$ converges for $Re(s) > a$ and*

$$\mathcal{L}\{f \star g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}.$$

Proof: The strategy of the proof is to express $\mathcal{L}\{f \star g\}$ as an iterated integral and then to change the order of integration.

$$\begin{aligned} \mathcal{L}\{f \star g\} &= \int_0^\infty e^{-st}(f \star g)(t)dt \\ &= \int_0^\infty e^{-st} \int_0^t f(t-\tau)g(\tau)d\tau dt \\ &= \iint_D e^{-st} f(t-\tau)g(\tau)dA. \end{aligned}$$

Here, the double integral is over the wedge shaped region D in the $t\tau$-plane given by the inequalities $0 \leq \tau < t$ and $t > 0$. By changing the order of integration, we get

$$\mathcal{L}\{f \star g\} = \int_0^\infty \int_\tau^\infty e^{-st} f(t-\tau)g(\tau)dt d\tau.$$

Now, we will make the change of variables $\eta = t - \tau$ in the inner integral. After this change of variables, the integral is over the first quadrant of the $\tau\eta$-plane.

$$\begin{aligned}
\mathcal{L}\{f \star g\} &= \int_0^\infty \int_0^\infty e^{-s(\tau+\eta)} f(\eta) g(\tau) d\eta d\tau \\
&= \left(\int_0^\infty e^{-s\eta} f(\eta) d\eta \right) \left(\int_0^\infty e^{-s\tau} g(\tau) d\tau \right) \\
&= \mathcal{L}\{f\}\mathcal{L}\{g\}.
\end{aligned}$$

□

Example 2.2 *Suppose that $y(t)$ is the solution of the initial value problem*

$$y'' + 5y' + 4y = g(t), \quad y(0) = y'(0) = 0.$$

Express $y(t)$ in terms of $g(t)$.

Solution: *Take Laplace transforms of both sides of the equation:*

$$\begin{aligned}
\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{g(t)\} \\
(s^2 + 5s + 4)Y(s) &= \mathcal{L}\{g(t)\} \\
Y(s) &= \frac{1}{s^2 + 5s + 4}\mathcal{L}\{g(t)\}
\end{aligned}$$

At this point, let us find the inverse Laplace transform of $(s^2 + 5s + 4)^{-1}$. First, decompose it into partial fractions and then take inverse Laplace transforms:

$$\begin{aligned}
\frac{1}{s^2 + 5s + 4} &= \frac{1/3}{s+1} - \frac{1/3}{s+4} \\
\mathcal{L}^{-1}\{\frac{1}{s^2 + 5s + 4}\} &= \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} = f(t).
\end{aligned}$$

Now, using the convolution theorem, the inverse Laplace transform of $Y(s) = \mathcal{L}\{f\}\mathcal{L}\{g\}$ must be $f \star g$. Therefore,

$$y(t) = (f \star g)(t) = \int_0^t \left(\frac{1}{3}e^{-(t-\tau)} - \frac{1}{3}e^{-4(t-\tau)} \right) g(\tau) d\tau.$$