ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

ME 208 DYNAMICS

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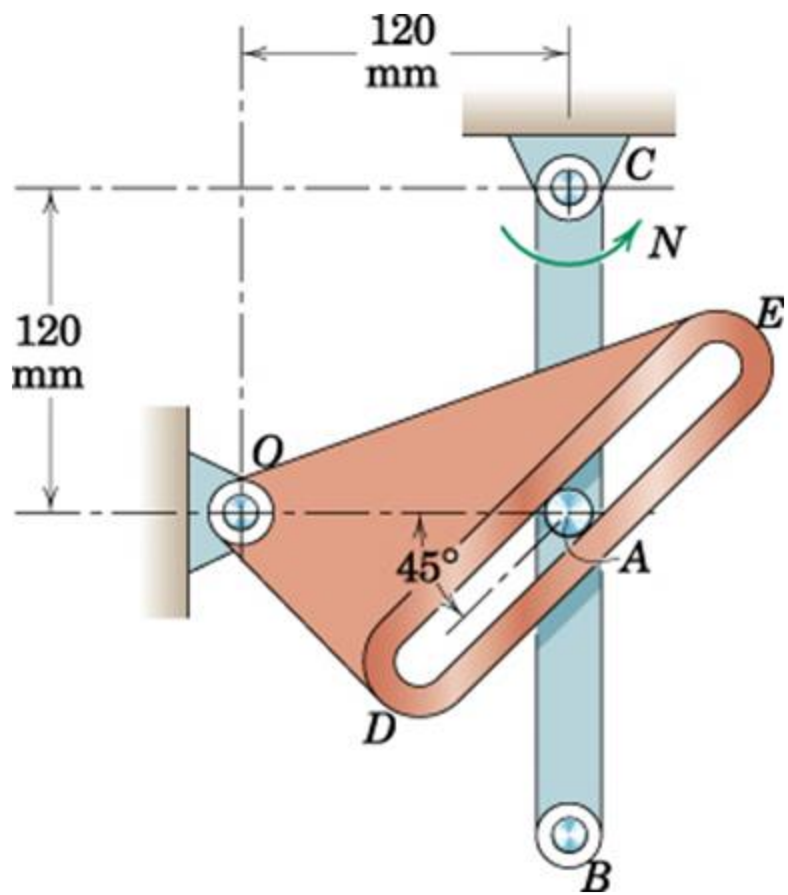
5/163 (4th), 5/168 (5th), 5/178 (6th), 5/176 (7th), 5/179 (8th)

For the instant represented, link CB is rotating counterclockwise at a constant rate N = 4 rad/s, and its pin A causes a clockwise rotation of the slotted member ODE. Determine the angular velocity ω and angular acceleration α of ODE for this instant.



Assume a point P fixed on body ODE instantly coincident with A.

$$\vec{v}_P = \vec{v}_A + \vec{v}_{P/A}$$

$$\vec{v}_P = \vec{\omega}_{ODE} \times \vec{r}_{P/O} = \omega_{ODE}\hat{k} \times 0.12\hat{\imath} = 0.12\omega_{ODE}\hat{\jmath}$$

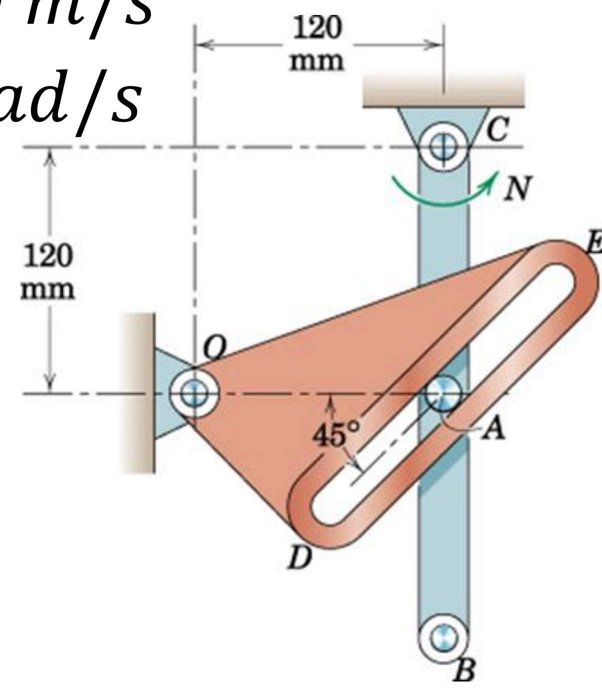
$$\vec{v}_A = N\hat{k} \times \vec{r}_{A/C} = 4\hat{k} \times -0.12\hat{\jmath} = 0.48\hat{\imath}\, m/s$$

$$\vec{v}_{P/A} = v_{P/A}(cos45°\hat{\imath} + sin45°\hat{\jmath})$$

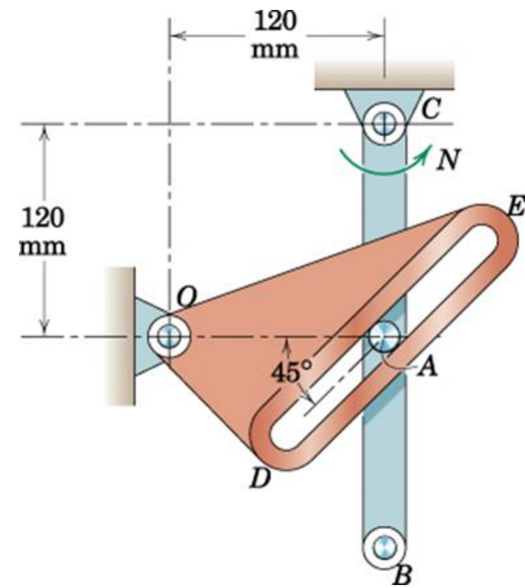
$$0.12\omega_{ODE}\hat{\jmath} = 0.48\hat{\imath} + v_{P/A}(cos45°\hat{\imath} + sin45°\hat{\jmath})$$

$$\hat{\imath}:\ 0 = 0.48 + v_{P/A}cos45°, v_{P/A} = -0.679\, m/s$$

$$\hat{\jmath}:\ 0.12\omega_{ODE} = v_{P/A}sin45°, \omega_{ODE} = -4\, rad/s$$



Assume a point P fixed on body ODE instantly coincident with A.



$\vec{a}_P = \vec{a}_A + \vec{a}_{P/A}$

$\vec{a}_P = \vec{a}_{P_t} + \vec{a}_{P_n}$

$\vec{a}_P = \vec{\alpha}_{ODE} \times \vec{r}_{P/O} - \omega_{ODE}{}^2 \vec{r}_{P/O}$

$\vec{a}_P = \alpha_{ODE} \hat{k} \times 0.12\hat{\imath} - 4^2 * 0.12\hat{\imath}$

$\vec{a}_P = (-1.920\hat{\imath} + 0.12\alpha_{ODE}\hat{\jmath})\ m/s^2$

$\vec{a}_A = \vec{a}_{A_t} + \vec{a}_{A_n} = \vec{\alpha}_{CA} \times \vec{r}_{A/C} - N^2 \vec{r}_{A/C}$

$\vec{a}_A = \vec{0} \times -0.12\hat{\jmath} - 4^2 * -0.12\hat{\jmath} = 1.920\hat{\jmath}\ m/s^2$

$\vec{a}_{P/A} = 2\vec{\omega}_{ODE} \times \vec{v}_{P/A} + \vec{a}_{rel}$

$\vec{a}_{P/A} = 2 * -4\hat{k} \times -0.679(cos45°\hat{\imath} + sin45°\hat{\jmath}) + a_{rel}(cos45°\hat{\imath} + sin45°\hat{\jmath})$

$\vec{a}_{P/A} = 3.84(-\hat{\imath} + \hat{\jmath}) + a_{rel}(cos45°\hat{\imath} + sin45°\hat{\jmath})$

$(-1.92\hat{\imath} + 0.12\alpha_{ODE}\hat{\jmath}) = 1.92\hat{\jmath} + 3.84(-\hat{\imath} + \hat{\jmath}) + a_{rel}(cos45°\hat{\imath} + sin45°\hat{\jmath})$

$\hat{\imath}$: $-1.92 = -3.84 + a_{rel} cos45°$

$a_{rel} = 2.72\ m/s^2$

$\hat{\jmath}$: $0.12\alpha_{ODE} = 1.92 + 3.84 + 2.72 sin45°$

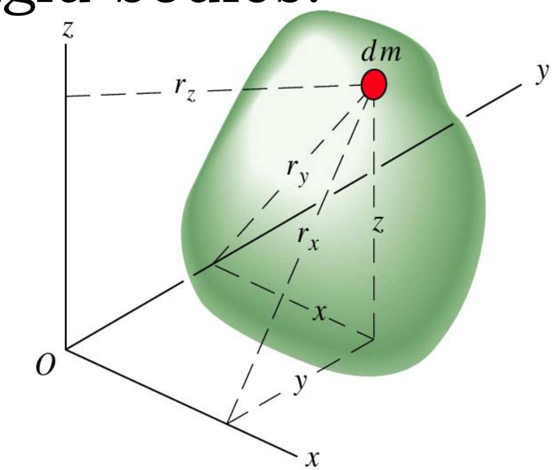
$\alpha_{ODE} = 64.0\ rad/s^2$

Plane Kinetics of Rigid Bodies

Chapter 4: Kinetics of Systems of Particles

4/1 Introduction

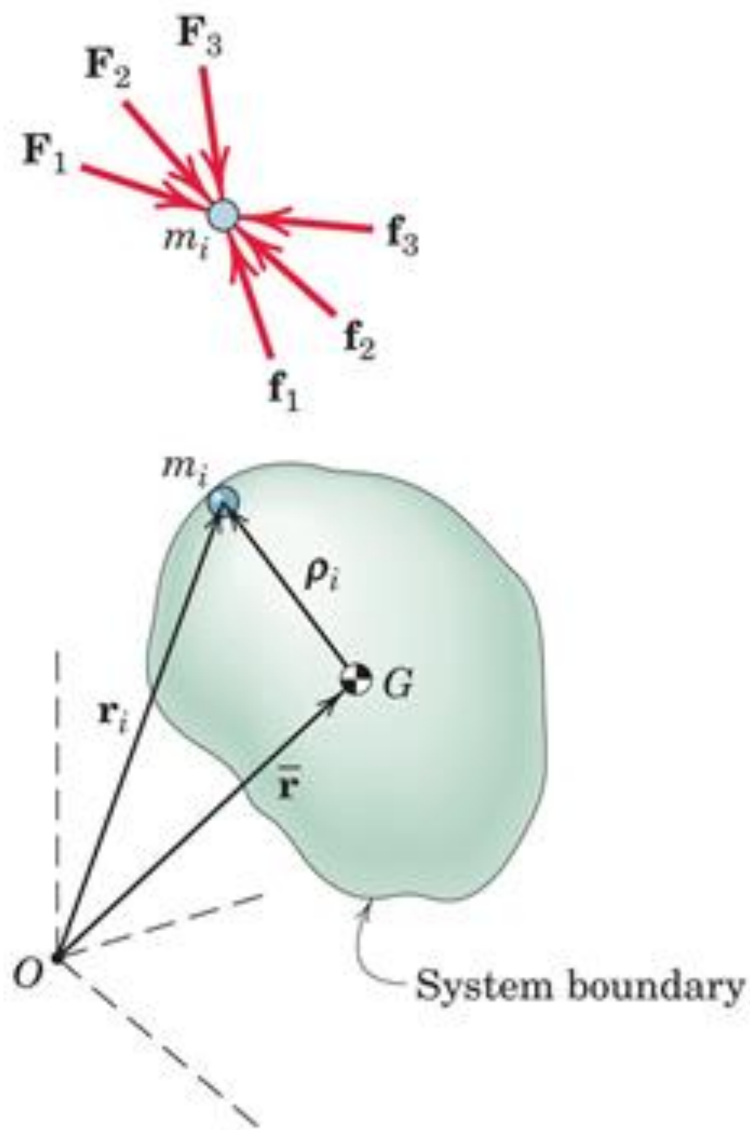
Since a rigid body is *assumed to be* a continuous collection of infinitely many particles of infinitesimal mass (*continuum approximation*) with no change in the relative positions of particles, the kinetic equations are derived using a general system of particles (rigid or deformable or even flowing) in our textbook. This approach is very general and when the rigidity condition is imposed the equations will simplify to what we will be using in Chapter 6 for kinetics of rigid bodies.



4/2 Generalized Newton’s Second Law

For a system of particles in a volume, let $\vec{F}_i$ be the resultant of forces acting on a particle m_i due to sources **external** to the system boundary whereas $\vec{f}_i$ be the resultant of forces on m_i due to sources **within** the system boundary. Newton’s second law for the particle i is then:

$$\vec{F}_i + \vec{f}_i = m_i\vec{a}_i$$



Now recall the definition of mass center (center of mass) from statics:

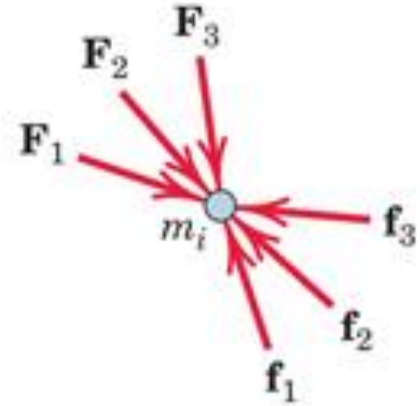
$$m\vec{r}_G = \sum_{i=1}^{n} m_i \vec{r}_i \left(= \int_m \vec{r}\, dm \right)$$

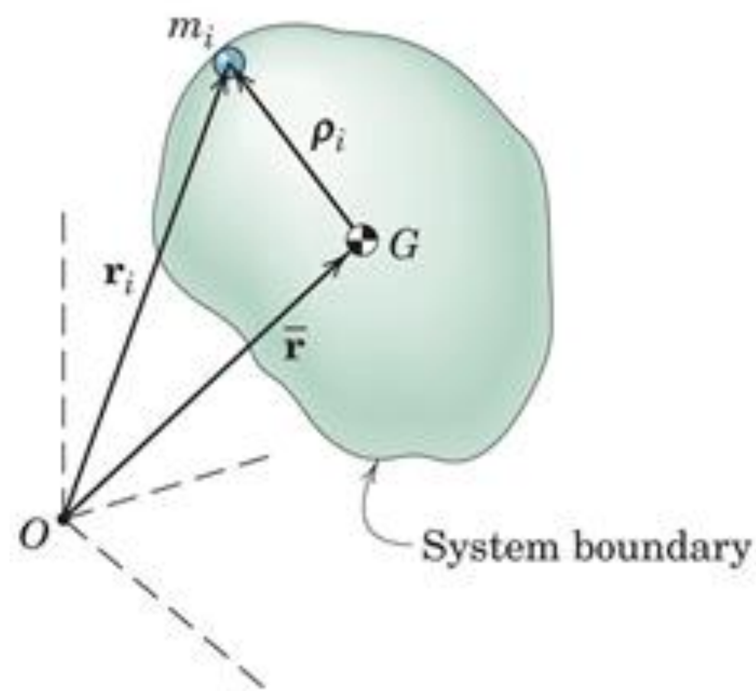
where

$$m = \sum_{i=1}^{n} m_i \left(= \int_m dm \right)$$

Second time derivative of this equation yields

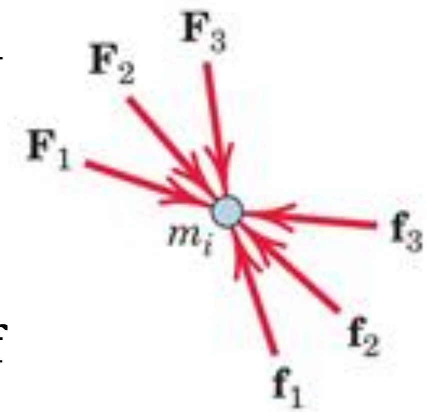
$$m\ddot{\vec{r}}_G = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} m_i \vec{a}_i$$





$$\sum_{i=1}^{n}(\vec{F}_i + \vec{f}_i) = \sum_{i=1}^{n} m_i\vec{a}_i = m\ddot{\vec{r}}_G = m\vec{a}_G$$

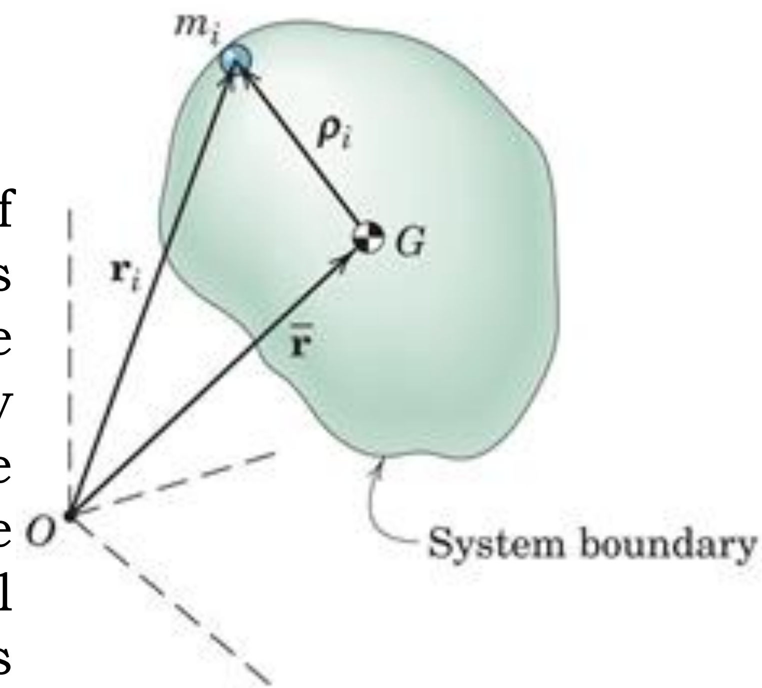
However when summed up within the system boundary, due to Newton's third law, *the action-reaction principle*,



$$\sum_{i=1}^{n} \vec{f}_i = \vec{0}$$

This leads to the principle of motion of center of mass:

$$\sum_{i=1}^{n} \vec{F}_i = m\vec{a}_G$$



which states that the acceleration of center of mass of the system of particles is in the same direction with the resultant ***external*** force and is inversely proportional with the total mass of the system of particles. Please note that the line of action of resultant of external forces *need not* pass through the mass center, G.

$$\sum_{i=1}^{n} \vec{F}_i = m\vec{a}_G$$

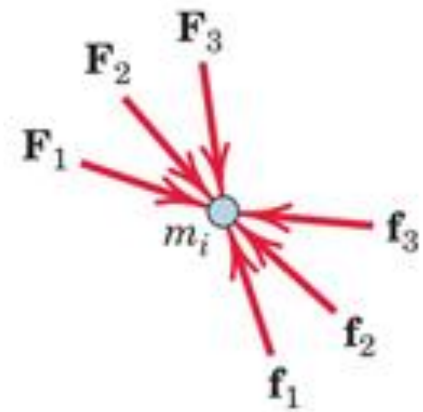
This vector equation may be resolved in any convenient coordinate system:

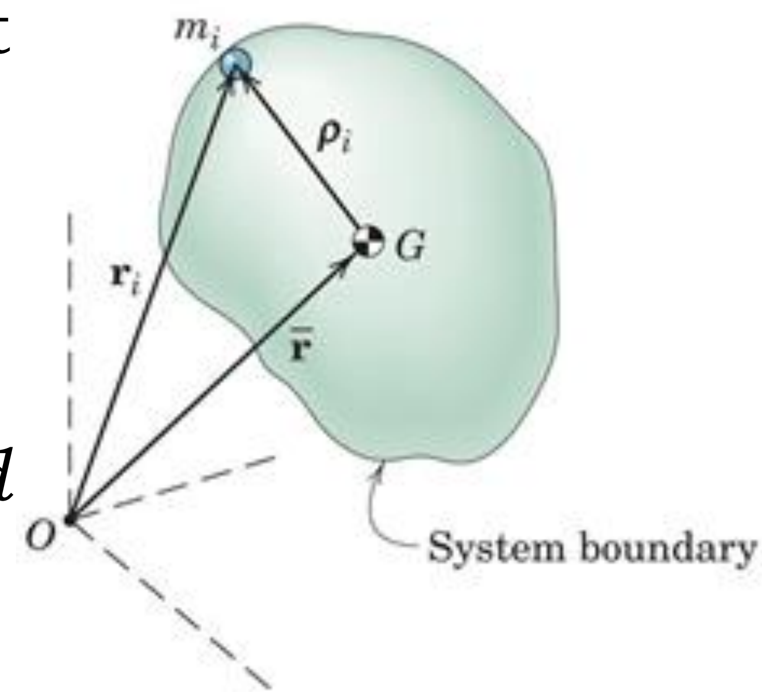
x-y,

n-t,

r-θ.

This is sufficient for rigid bodies.





4/3 Work-Energy

For a particle of mass m_i the work-energy relation:

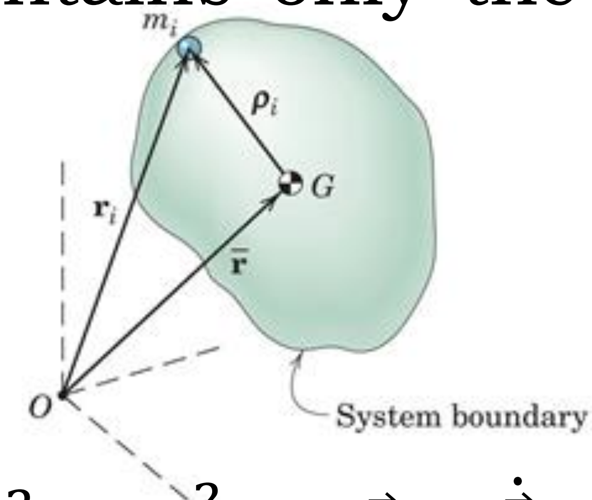
$U_i = \Delta T_i + \Delta V_{g_i} + \Delta V_{e_i}$

The work done by internal forces cancel each other because for a rigid body the action and reaction pairs have identical displacements therefore the work done only by the external forces need to be considered during summation. However for non-rigid bodies displacements of action reaction pairs may be different causing storage of elastic potential energy or dissipation of mechanical energy which we will not discuss in this course.

$$\sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \Delta T_i + \sum_{i=1}^{n} \Delta V_{g_i} + \sum_{i=1}^{n} \Delta V_{e_i}$$

For the system

$U_{1\to 2} = \Delta T + \Delta V_g + \Delta V_e$ where $U_{1\to 2}$ contains only the work done by external forces.



$$\Delta T = \sum_{i=1}^{n} \frac{1}{2} m_i {v_i}^2$$

$$\vec{v}_i = \vec{v}_G + \dot{\vec{\rho}}_i$$

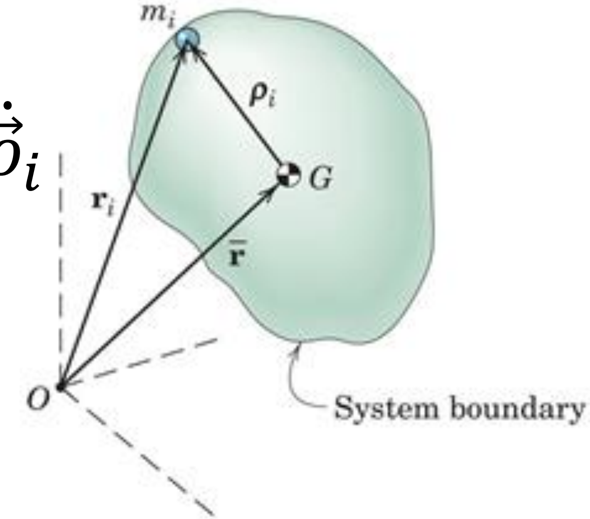
$${v_i}^2 = \vec{v}_i \cdot \vec{v}_i = \left(\vec{v}_G + \dot{\vec{\rho}}_i\right) \cdot \left(\vec{v}_G + \dot{\vec{\rho}}_i\right) = {v_G}^2 + {\dot{\rho}_i}^2 + 2\vec{v}_G \cdot \dot{\vec{\rho}}_i$$

Substitution into kinetic energy expression yields

$$\Delta T = \sum_{i=1}^{n} \frac{1}{2} m_i {v_G}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i {\dot{\rho}_i}^2 + \sum_{i=1}^{n} m_i \vec{v}_G \cdot \dot{\vec{\rho}}_i$$

$$\Delta T = \frac{1}{2} m {v_G}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i {\dot{\rho}_i}^2 + \vec{v}_G \cdot \sum_{i=1}^{n} m_i \dot{\vec{\rho}}_i$$

$$\Delta T = \frac{1}{2} m {v_G}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i {\dot{\rho}_i}^2 + \vec{v}_G \cdot \sum_{i=1}^{n} m_i \dot{\vec{\rho}}_i$$



but

$$\sum_{i=1}^{n} m_i \vec{\rho}_i = \vec{0}$$

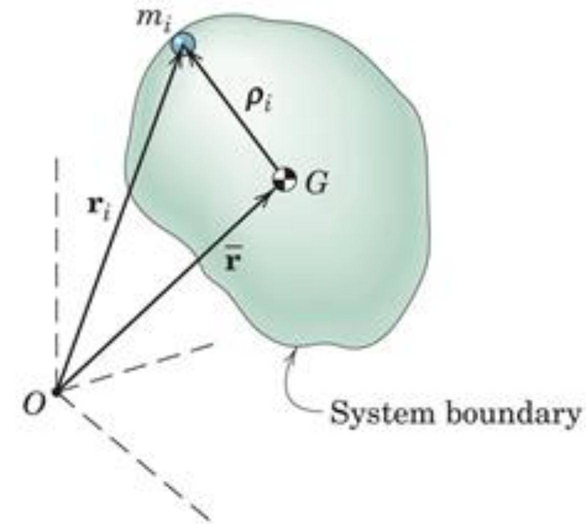
(definition of mass center so its time derivative is zero too!)

$$\Delta T = \frac{1}{2} m {v_G}^2 + \sum_{i=1}^{n} \frac{1}{2} m_i {\dot{\rho}_i}^2$$

The first term is translational kinetic energy of the mass center, G, the second term is due to relative motion of particles with respect to the mass center, G.

4/4 Impulse Momentum

Linear Momentum



$$\vec{G}_i = m_i \vec{v}_i$$

$$\vec{G} = \sum_{i=1}^{n} \vec{G}_i = \sum_{i=1}^{n} m_i \vec{v}_i = \sum_{i=1}^{n} m_i \left(\vec{v}_G + \dot{\vec{\rho}}_i\right)$$

$$\vec{G} = \sum_{i=1}^{n} m_i \vec{v}_G + \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{\rho}_i = \vec{v}_G \sum_{i=1}^{n} m_i + \vec{0} = m \vec{v}_G$$

$$\dot{\vec{G}} = \frac{d\vec{G}}{dt} = \frac{d}{dt}(m\vec{v}_G) = m\vec{a}_G = \sum \vec{F}$$

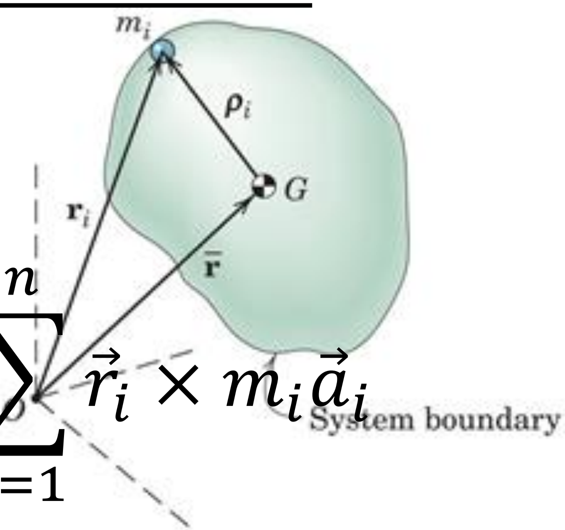
This is valid for constant mass ($\dot{m} = 0$) systems.

Angular Momentum

- <u>Angular Momentum about a Fixed Point O</u>

$$\vec{H}_O = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i$$

$$\dot{\vec{H}}_O = \sum_{i=1}^{n} \dot{\vec{r}}_i \times m_i \vec{v}_i + \sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{v}}_i = \vec{0} + \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{a}_i$$

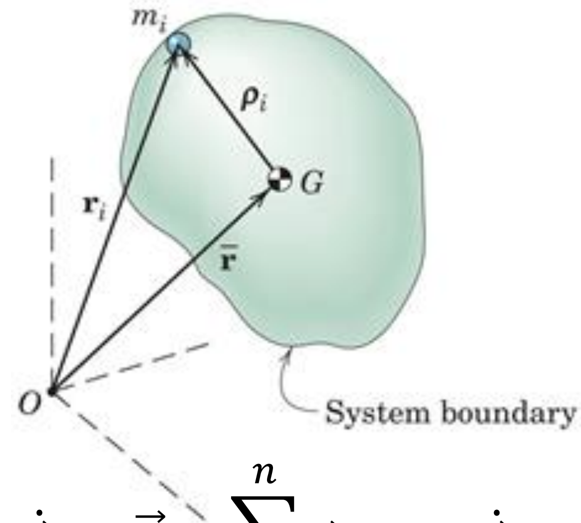


$$\dot{\vec{H}}_O = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = \sum_{i=1}^{n} \vec{M}_O$$

According to Varignon's principle sum of moments of forces is equal to the moment of the resultant of the forces.

Angular Momentum

- Angular Momentum about Mass Center G



$$\vec{H}_G = \sum_{i=1}^{n} \vec{\rho}_i \times m_i \vec{v}_i \,, \vec{v}_i = \vec{v}_G + \dot{\vec{\rho}}_i$$

$$\vec{H}_G = \sum_{i=1}^{n} \vec{\rho}_i \times m_i \left(\vec{v}_G + \dot{\vec{\rho}}_i\right) = -\vec{v}_G \times \sum_{i=1}^{n} m_i \vec{\rho}_i + \sum_{i=1}^{n} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i = \vec{0} + \sum_{i=1}^{n} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

$$\dot{\vec{H}}_G = \sum_{i=1}^{n} \dot{\vec{\rho}}_i \times m_i \dot{\vec{\rho}}_i + \sum_{i=1}^{n} \vec{\rho}_i \times m_i \ddot{\vec{\rho}}_i = \vec{0} + \sum_{i=1}^{n} \vec{\rho}_i \times m_i \ddot{\vec{\rho}}_i$$

$$\vec{a}_i = \vec{a}_G + \ddot{\vec{\rho}}_i, \ddot{\vec{\rho}}_i = \vec{a}_i - \vec{a}_G$$

$$\dot{\vec{H}}_G = \sum_{i=1}^{n} \vec{\rho}_i \times m_i (\vec{a}_i - \vec{a}_G) = \sum_{i=1}^{n} \vec{\rho}_i \times m_i \vec{a}_i + \vec{a}_G \times \sum_{i=1}^{n} \vec{\rho}_i m_i = \sum_{i=1}^{n} \vec{\rho}_i \times \vec{F}_i + \vec{0}$$

According to Varignon’s theorem

$$\dot{\vec{H}}_G = \sum \vec{M}_G$$

Angular Momentum

- <u>Angular Momentum about an Arbitrary Point P</u>

$$\vec{H}_P = \sum_{i=1}^{n} \overrightarrow{\rho'}_i \times m_i \vec{v}_i$$

$$\overrightarrow{\rho'}_i = \vec{\rho}_G + \vec{\rho}_i$$

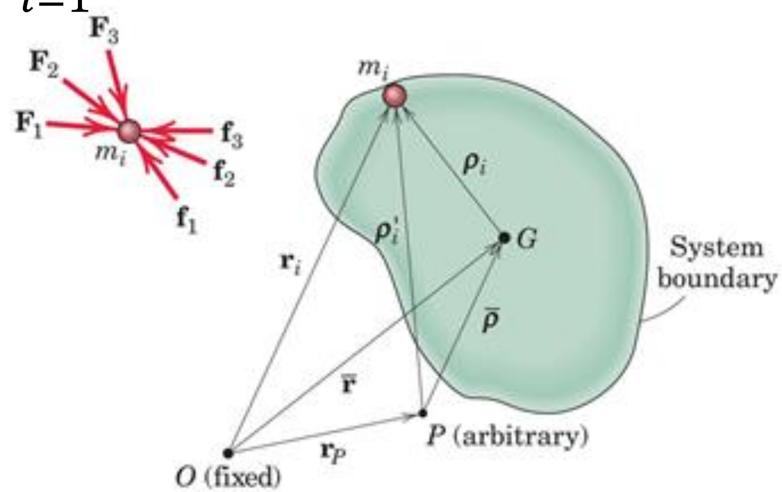
$$\vec{H}_P = \sum_{i=1}^{n} (\vec{\rho}_G + \vec{\rho}_i) \times m_i \vec{v}_i = \vec{\rho}_G \times \sum_{i=1}^{n} m_i \vec{v}_i + \sum_{i=1}^{n} \vec{\rho}_i \times m_i \vec{v}_i = \vec{\rho}_G \times \vec{G} + \vec{H}_G$$

$$\vec{H}_P = \vec{H}_G + \vec{\rho}_G \times \vec{G}$$

One may write

$$\sum \vec{M}_P = \sum \vec{M}_G + \vec{\rho}_G \times \sum \vec{F}$$

$$\sum \vec{M}_P = \dot{\vec{H}}_G + \vec{\rho}_G \times m\vec{a}_G$$



Moment about an arbitrary point is time rate of angular momentum about mass center plus moment of $m\vec{a}_G$ about P.

4/5 Conservation of Energy and Momentum

- *Conservation of Energy*

A conservative system does not lose mechanical energy due to internal friction or other types of inelastic forces. If no work is done on a conservative system by external forces then

$U_{1\to2} = 0 = \Delta T + \Delta V_g + \Delta V_e$

- *Conservation of Momentum*

If for an interval of time

$$\sum \vec{F} = \vec{0}, \dot{\vec{G}} = \vec{0}, \Delta\vec{G} = \vec{0}$$

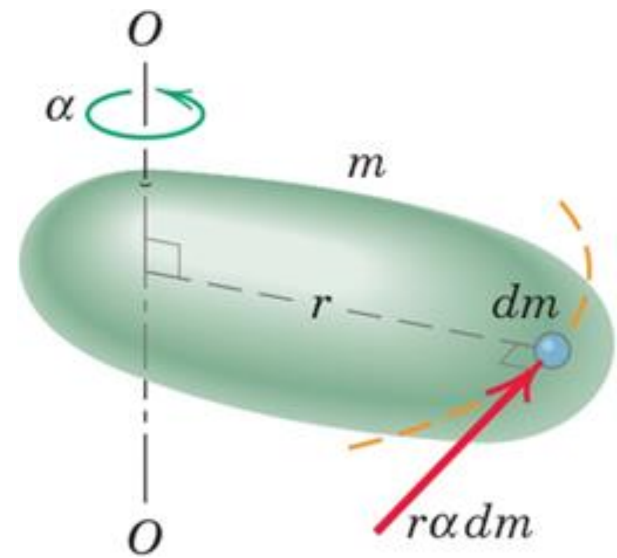
Again for an interval of time if

$$\sum \vec{M}_G = \vec{0}, \dot{\vec{H}}_G = \vec{0}, \Delta\vec{H}_G = \vec{0}$$

$$\sum \vec{M}_O = \vec{0}, \dot{\vec{H}}_O = \vec{0}, \Delta\vec{H}_O = \vec{0}$$

Appendix B: Mass Moment of Inertia

B/1 Mass Moment of Inertia about an Axis



$a_t = r\alpha$

$dF = r\alpha dm$

Moment of this force about O-O

$dM = rdF = r^2\alpha dm$

For all particles in the rigid body

$$M = \int_m dM = \int_m r^2\alpha dm = \alpha\int_m r^2 dm$$

Mass moment of inertia of the body about axis O-O is defined as

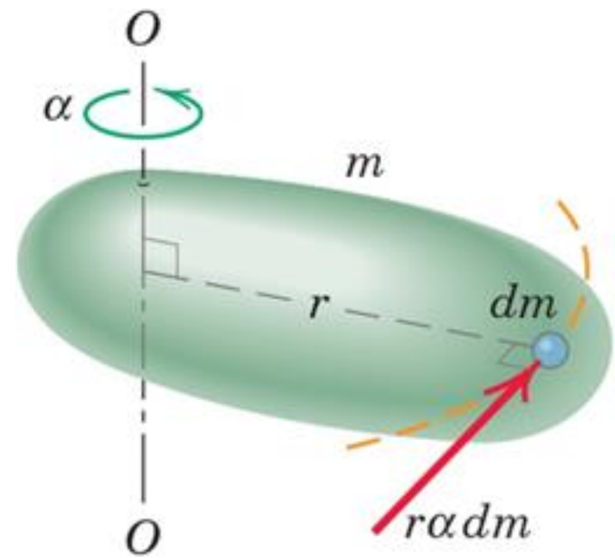
$$I_O \equiv \int_m r^2 dm \ \ [kg \cdot m^2]$$

Mass is a resistance to linear acceleration, mass moment of inertia is a resistance to angular acceleration.

For discrete system of particles

$$I_O = \sum_{i=1}^{n} {r_i}^2 m_i$$

For constant density (homogeneous) rigid bodies

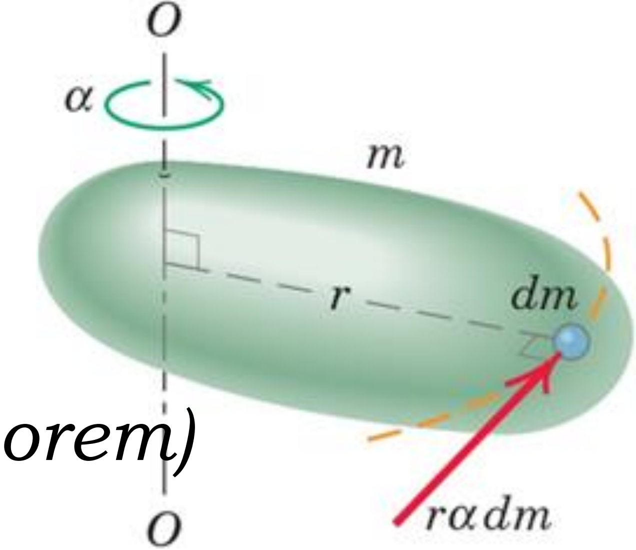


$$I_O = \rho \int_V r^2 dV$$

Radius of Gyration

$$k_x \equiv \sqrt{\frac{I_x}{m}}, I_x = {k_x}^2 m$$

Radius of gyration is a measure of mass distribution of a rigid body about the axis. A system with equivalent mass moment of inertia is a very thin ring of same mass and radius k_x.



Transfer of Axis (Parallel Axis Theorem)

$$I_x = I_G + m|gX|^2$$

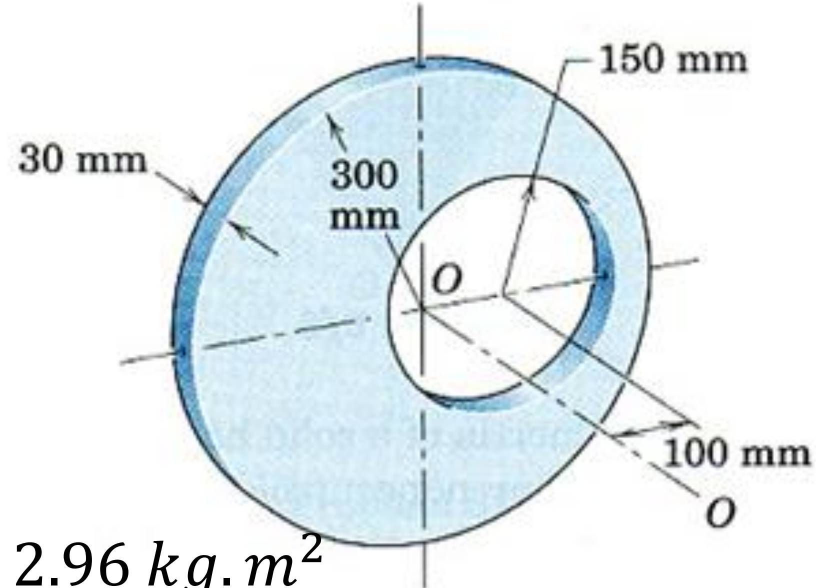
$$k_x^{\,2} = k_G^{\,2} + |gX|^2$$

Composite Bodies

Mass moment of inertia of a composite body about an axis is the sum of individual mass moments of each part about the same axis (which may be calculated utilizing parallel axis theorem if mass moment of inertia of each part is known about its mass center).

B/ 10 (4th), None (5th), B/ 14 (6th), None (7th), None (8th)

Calculate the mass moment of inertia about the axis O-O for the steel disk with the hole.



$$I_O = I_{O_{solid}} - I_{O_{hole}}$$

For thin disks

$$I_G = \frac{1}{2}mr^2$$

$$I_{O_{solid}} = I_G = \frac{1}{2}mr^2 = \frac{1}{2}\rho\pi {r_d}^2 t {r_d}^2 = 2.96\ kg.m^2$$

$$I_{O_{hole}} = I_G + md^2 = \frac{1}{2}mr^2 + md^2 = \frac{1}{2}\rho\pi {r_h}^2 t {r_h}^2 + \rho\pi {r_h}^2 d^2$$

$$= 0.348\ kg.m^2$$

$$I_O = 2.96 - 0.348 = 2.61\ kg.m^2$$

$$k_O = \sqrt{\frac{I_O}{m}} = \sqrt{\frac{2.61}{\rho\pi {r_d}^2 t - \rho\pi {r_h}^2 t}} = 0.230\ m$$

Chapter 6: Plane Kinetics of Rigid Bodies

6/1 Introduction

In this chapter we will deal with relations among external forces and moments, and, translational and rotational motions of rigid bodies in plane motion.

We will write two force and one moment equation (*or equivalent*) for the plane motion of rigid bodies.

The equations derived in Chapter 4 will be simplified for a rigid body and used. Kinematic relations developed in Chapters 2 and 5 will be utilized.

Drawing correct free body diagrams is essential in application of **Force-Mass-Acceleration** method. The other two methods, similar to kinetics of particles are **Work-Energy** and **Impulse-Momentum**.

A. Direct Application of Newton's Second Law – Force Mass Acceleration Method for a Rigid Body

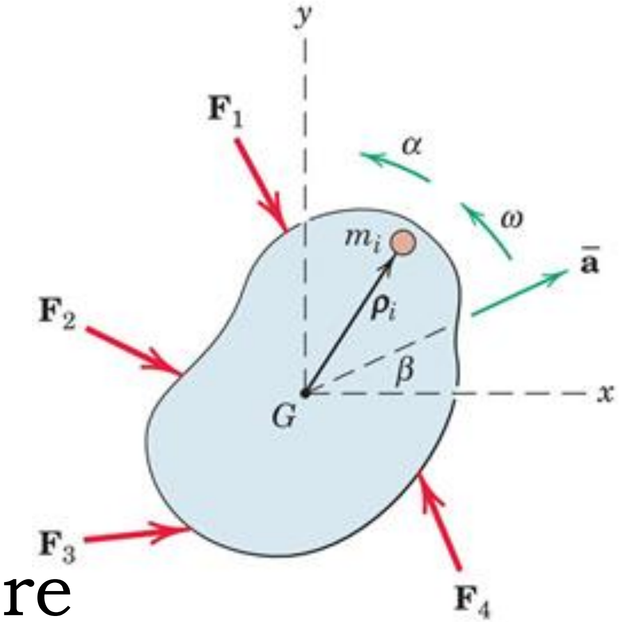
6/2 General Equations of Motion

$$\sum \vec{F} = \dot{\vec{G}} = m\vec{a}_G$$

$$\sum \overrightarrow{M}_G = \dot{\vec{H}}_G = I_G\vec{\alpha}$$

These are known as Euler's first and second laws.
By using statics information one may replace the forces on a rigid body by a single resultant force passing through mass center and a couple moment. The equivalent force causes linear acceleration of the mass center in the direction of force, the couple moment causes angular acceleration about the axis of the couple moment.

Plane Motion Equations



$$\sum \vec{F} = m\vec{a}_G$$

$$\vec{H}_G = \sum_{i=1}^{n} \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i = \int_m \vec{\rho} \times \dot{\vec{\rho}} dm$$

For a rigid body $|\vec{\rho}_i| = const$ therefore

$$\dot{\vec{\rho}} = \vec{\omega} \times \vec{\rho}$$

$$\vec{\rho} \times \dot{\vec{\rho}} = \vec{\rho} \times \vec{\omega} \times \vec{\rho} = -\vec{\rho} \times (\vec{\rho} \times \vec{\omega}) = \rho^2 \vec{\omega}$$

$$\vec{H}_G = \int_m \rho^2 \vec{\omega} dm = \vec{\omega} \int_m \rho^2 dm = \vec{\omega} I_G$$

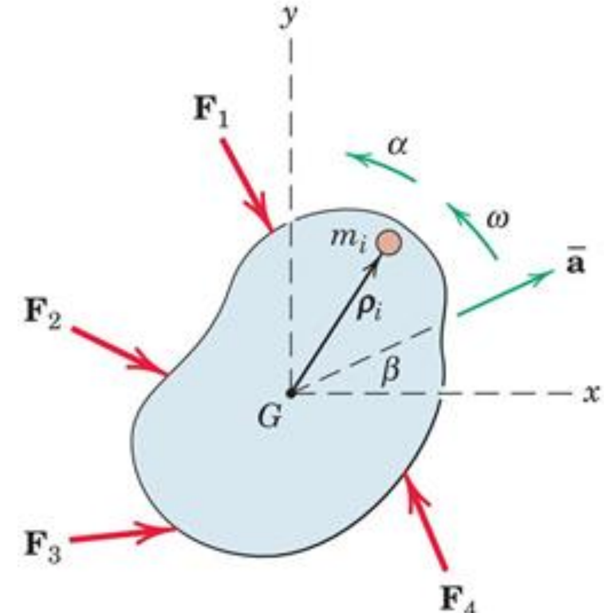
For a rigid body I_G is constant so

$$\dot{\vec{H}}_G = I_G \dot{\vec{\omega}} = I_G \vec{\alpha}$$

$$\sum \vec{F} = m\vec{a}_G \,, \sum \vec{M}_G = I_G \vec{\alpha}$$

$$\sum \vec{F} = m\vec{a}_G$$

$$\sum \overrightarrow{M}_G = I_G \vec{\alpha}$$



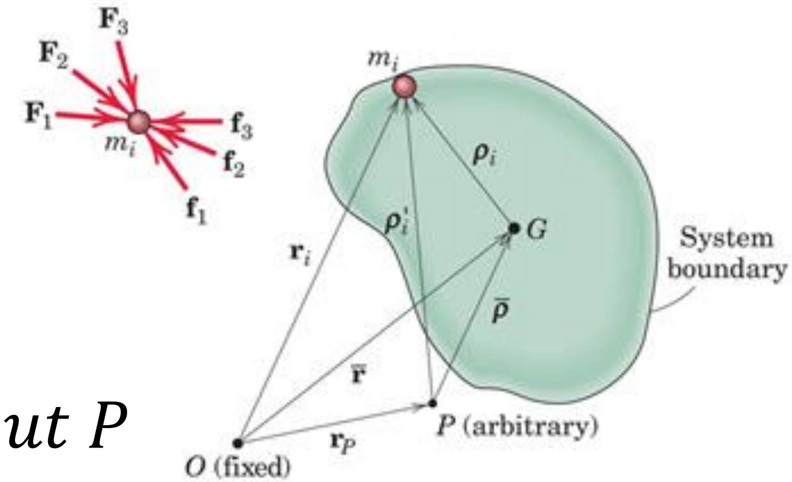
Euler's laws of motion, generalization of Newton's second law for particles to rigid bodies approximately 50 years after Newton.

For plane motion the force equation may be resolved in x-y, n-t or r-θ coordinates whichever is suitable. For moment equation it is always normal to the plane of motion therefore can be expressed in scalar form as CCW or CW.

The moment equation has an alternative derivation yielding the same result. Please go over it in the textbook.

Alternative Moment Equation

Sometimes it may be more convenient to take moment about another point rather than the mass center G. In that case



$$\sum \vec{M}_P = \dot{\vec{H}}_G + \vec{\rho}_G \times m\vec{a}_G, \vec{\rho}_G = \overrightarrow{PG}$$

This equation can be written as

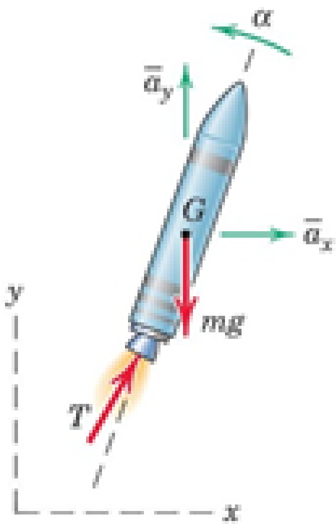
$$\sum M_P = I_G\alpha + Moment\ of\ m\vec{a}_G\ about\ P$$

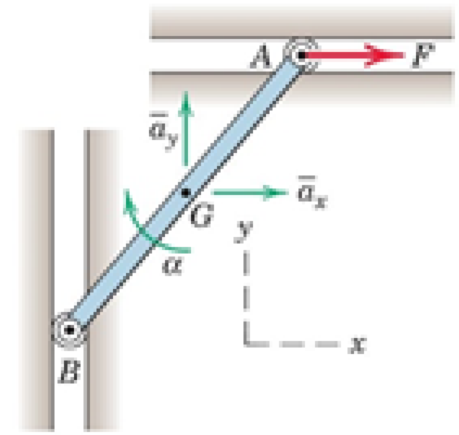
If point P is a fixed point (like the axis of rotation) then

$$\sum M_O = I_G\alpha + Moment\ of\ m\vec{a}_G\ about\ P, a_{G_t} = |OG|\alpha,$$

$$Moment\ of\ m\vec{a}_G\ about\ P = |OG|^2\alpha$$

$$\sum M_O = I_G\alpha + |OG|^2 m\alpha = (I_G + m|OG|^2)\alpha = I_O\alpha$$





In unconstrained motion the two components of acceleration of the mass center and angular acceleration are *independent* of each other as in the case of a rocket.

In constrained motion due to kinematic restrictions *there are relations* among two components of the acceleration of mass center and the angular acceleration of the body. Therefore these kinematic constraint equations have to be determined using methods developed in Chapter 5. There are also reaction forces due to constraints in the direction of restricted motions which should be included in the free body diagram.

Analysis Procedure

Kinematics: Determine $\vec{v}_G$, $\vec{a}_G$, ω and α (or the kinematic relations among them) if possible.

Diagrams: Draw proper free body and kinetic diagrams.

Equations of motion: Any force or acceleration in the direction of positive coordinate is positive. Count the number of available independent equations and number of unknowns to be determined.