2. Kinematic Analysis

Representation of Plane Vectors by Complex Numbers

Review of Complex Numbers

- $i$ is an operator that rotates a vector 180°.
- The unit imaginary number $i \ (i^2 = -1)$ is an operator that rotates a vector 90° counter clockwise.
- Please note that twice 90° counter clockwise which is 180° is $i^2 = -1$.
- $-i = i$ rotates a vector 90° clockwise.
- The complex plane (also called Gauss-Argand plane) is analogous to the two dimensional Cartesian coordinates $(x + Rr, y + IIm)$

• $r = x + iy = x + i y$

• Complex numbers can be represented in polar form as well:
  - $z = x + iy = r (\cos \theta + i \sin \theta)$
  - $r = \sqrt{x^2 + y^2}, \ \theta = \arctan(y/x)$
  - Euler's identity $e^i = \cos + i \sin$ so $e = r e^i$
  - Multiplication of a real number, $a$, with $r e^i$ rotates the real number $a$ counter clockwise

• Vector addition and addition of two complex numbers are analogous.
• In complex numbers we do not need cross product or out of plane angular vectors like $d$ and $d!$

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Coincident Points

Permanently Coincident Points: Two points on two different rigid bodies are coincident for all possible positions of the mechanism.

Typically the points on the axis of a revolute joint which connects two rigid bodies are permanently coincident.

Instantly Coincident Points: Two points on two different rigid bodies are coincident only for the current position of the mechanism.

Typically the instant center of zero velocity of a link does not have a fixed location with respect to another link including the fixed link (and also it does not have a fixed location relative to its own body).

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Vector Loops of Mechanisms (Constraint Equations)

Suppose we select Point B:

$$ A_0A + AB_1 = A_0B_0 + B_0B_4 $$

This vector equation forces $B_3$ and $B_4$ be a permanently coincident point, the revolute joint between links 3 and 4. It is called loop closure equation and is the constraint equation of the four-bar mechanism.

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Kinematics of Rigid Body in Plane Motion

- Motion of a rigid body in plane can be described fully by the motion of two points on the plane.
- Rigidity condition ensures that the velocity components of the two selected points along the line connecting the two points should be equal.
- It is sufficient to represent a rigid body (which may be considered as an infinite plane) by the two representative points and the line connecting them.

2. Kinematic Analysis

Vector Loops of Mechanisms (Constraint Equations)

$$ A_0A + AB_1 = A_0B_0 + B_0B_4 $$

This vector equation can be written using complex numbers as:

$$ a_1 e^{i \theta_1} + a_4 e^{i \theta_4} = a_1 + a_4 e^{i \theta_4} $$

This is a complex number in three unknowns, $\theta_1$, $\theta_2$, and $\theta_4$. If one of those variables (recall $P=1$ for a four-bar) is known the other two can be determined (as we will see later).
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Vector Loops of Mechanisms (Constraint Equations)
Suppose we select point A this time:
$$\vec{A}_0\vec{A}_2 = \vec{A}_0\vec{A}_0 + \vec{B}_0\vec{B} + \vec{B}_0\vec{A}_3$$
This vector equation forces A₂ and A₃ to be a permanently coincident point, the revolute joint between links 2 and 3, A. It looks different from the previous equation where we disconnected revolute joint B and force it to be permanently coincident.
$$\vec{A}_0\vec{A} + \vec{B}_3 = \vec{A}_0\vec{B}_0 + \vec{B}_0\vec{A}_4$$

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Slider-Crank
$$\vec{A}_0\vec{A} + \vec{B}_1 = \vec{A}_0\vec{B}_4$$
This vector equation forces B₃ and B₁ to be a permanently coincident point, the revolute joint between links 3 and 4, B. This vector equation can be written using complex numbers as:
$$a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = ia_1 + s_{14}$$
This is a complex equation in three unknowns, $\theta_{12}$, $\theta_{13}$ and $s_{14}$. If one of those variables (recall F = 1 for a slider-crank) is known the other two can be determined.

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Inverted Slider-Crank
$$\vec{A}_0\vec{A}_2 = \vec{A}_0\vec{B}_0 + \vec{B}_3$$
This vector equation forces A₂ and A₃ to be a permanently coincident point, the revolute joint between links 2 and 3, A. This vector equation can be written using complex numbers as:
$$a_2e^{i\theta_{12}} = a_1 + a_4e^{i\theta_{14} + s_{43}e^{i\theta_{14} + \phi_4}}$$
This is a complex equation in three unknowns, $\theta_{12}$, $\theta_{14}$ and $s_{43}$. If one of those variables (recall F = 1 for an inverted slider-crank) is known the other two can be determined.

2. Kinematic Analysis

Vectors Allowed in a Loop Closure Equation
1. **Body Vectors**: They are the vectors that connect the two points on the *same* link. These vectors have a constant magnitude but the orientation may change.
2. **Translational Joint Variable Vectors**: They are the vectors between two links which are connected by a prismatic or cylinder in slot joint. These vectors are parallel to the relative sliding axis and have a variable magnitude. Direction may be variable as well.
3. **Zero Vectors**: They are the vectors connecting two permanently coincident points on two different links. They are *not* written!
2. Kinematic Analysis

**Inverted Slider-Crank**

\[
A_0A_1 + A_0A_2 = A_0B_0 + B_0B_0 + B_0Q_4 + Q_4A_2
\]

Zero Vectors

Body Vectors

**2. Kinematic Analysis**

**Multiloop Mechanisms**

Euler's polyhedron formula tells the number of independent loops as:

\[ L = j - \ell + 1 = 7 - 6 + 1 = 2 \]

Next question is which of the following five equations are independent?

1. \( A_0A + A_0B + BD_3 = A_0D_0 + D_0D_3 \) (1)
2. \( A_0A + A_0C + CE_4 = A_0D_0 + D_0E_2 \) (2)
3. \( CE + ED_3 = CB + BD_3 \) (3)
4. \( AB + BC + CA = 0 \) (4)
5. \( D_3D + DE + ED_3 = 0 \) (5)

**2. Kinematic Analysis**

**Multiloop Mechanisms**

Is there a way to find out independent loop closure equations?

**Yes, there is!**

1. Disconnect gear pair(s) (if any) and write the gear relations.
2. Disconnect as many revolute joints as necessary to eliminate all loops. (However no link should be totally disconnected!)
3. By connecting only one joint at a time (all others should be disconnected during this process) write the loop formed by connecting this joint.

**2. Kinematic Analysis**

**Multiloop Mechanisms**

There are many loops like:

1. \( A_0A + A_0B + BD_3 = A_0D_0 + D_0D_3 \) (1)
2. \( A_0A + A_0C + CE_4 = A_0D_0 + D_0E_2 \) (2)
3. \( CE + ED_3 = CB + BD_3 \) (3)
4. \( AB + BC + CA = 0 \) (4)
5. \( D_3D + DE + ED_3 = 0 \) (5)

Equations 4 & 5 are identities.

Equations 1, 2 and 3 are valid loop closure equations. However they are not all independent.

Is there a way to find out independent loop closure equations?