2. Kinematic Analysis

Representation of Plane Vectors by Complex Numbers

Review of Complex Numbers

- \( -1 \) is an operator that rotates a vector 180°.
- The unit imaginary number \( i (i^2 = -1) \) is an operator that rotates a vector 90°
  counterclockwise.
- Please note that twice 90° counter rotation which is 180° in \( i^2 = -1 \)
  \( -i = \frac{1}{i} \) rotates a vector 90° clockwise.
- The complex plane (also called Gauss-Argand plane) is analogous to the two dimensional Cartesian coordinates \( (x \rightarrow Re, y \rightarrow Im) \)
  \( \vec{r} = \vec{i} \rightarrow x + iy \rightarrow x + iy \)
- Complex numbers can be represented in polar form as well:
  \( z = x + iy = r (\cos \theta + i \sin \theta) \).
- Euler’s identity \( e^{i \theta} = \cos \theta + i \sin \theta \).
- Multiplication of a real number \( r \) with \( i \) rotates the real number \( \theta \) counter clockwise.
- Vector addition and addition of two complex numbers are analogous.
- In complex numbers we do not need cross product or out of plane angular vectors like \( \alpha \) and \( \delta \).

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Coincident Points

Permanently Coincident Points: Two points on two different rigid bodies are coincident for all possible positions of the mechanism.

Typically the points on the axis of a revolute joint which connects two rigid bodies are permanently coincident.

Instantly Coincident Points: Two points on two different rigid bodies are coincident only for the current position of the mechanism.

Typically the instant center of zero velocity of a link does not have a fixed location with respect to another link including the fixed link (and also it does not have a fixed location relative to its own body).

2. Kinematic Analysis

Vector Loops of Mechanisms (Constraint Equations)

Suppose we select point B:
\( \overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_3} + \overrightarrow{B_0B_4} \)

This vector equation forces \( B_3 \) and \( B_4 \) be a permanently coincident point, the revolute joint between links 3 and 4. B. It is called loop closure equation and is the constraint equation of the four-bar mechanism.

2. Kinematic Analysis

Kinematics of Rigid Body in Plane Motion

- Motion of a rigid body in plane can be described fully by the motion of two points on the plane.
- Rigidity condition ensures that the velocity components of the two selected points along the line connecting the two points should be equal.
- It is sufficient to represent a rigid body (which may be considered as an infinite plane) by the two representative points and the line connecting them.

2. Kinematic Analysis

Vector Loops of Mechanisms (Constraint Equations)

The three moving bodies are each represented by two points on them:
- Link 2: \( A_0A \)
- Link 3: \( AB \)
- Link 4: \( B_0B \)

Rather than searching for tedious geometric relations as we did in ME 208 Dynamics we will assume one of the permanently coincident points to be non-coincident and the constraint equation we will write is going to force these two points to be coincident. This will be a vector loop equation and the constraint equation for the mechanism.

This vector equation can be written using complex numbers as:
\( a_2e^{i\theta_{12}} + a_4e^{i\theta_{14}} = a_1 + a_4e^{i\theta_{14}} \)

This is a complex equation in three real unknowns, \( \theta_{12}, \theta_{13} \) and \( \theta_{14} \). If one of those variables (recall \( P = 1 \) for a four-bar) is known the other two can be determined (as we will see later!).

2. Kinematic Analysis

Vector Loops of Mechanisms (Constraint Equations)

\( \overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_3} + \overrightarrow{B_0B_4} \)

This vector equation can be written using complex numbers as:
\( a_2e^{i\theta_{12}} + a_4e^{i\theta_{14}} = a_1 + a_4e^{i\theta_{14}} \)

This is a complex equation in three real unknowns, \( \theta_{12}, \theta_{13} \) and \( \theta_{14} \). If one of those variables (recall \( P = 1 \) for a four-bar) is known the other two can be determined (as we will see later!).
2. Kinematic Analysis

Vector Loops of Mechanisms (Constraint Equations)
Suppose we select point A this time:
\[ A_0A_2 = A_0B_0 + B_0B + BA_3 \]
This vector equation forces A_3 and A_2 to be a permanently coincident point, the revolute joint between links 2 and 3, A. It looks different from the previous equation when we disconnected revolute joint B and force it to be permanently coincident.
\[ A_0A + AB_3 = A_0B_0 + B_0B + BA_3 \]

2. Kinematic Analysis

Slider-Crank
This vector equation forces B_3 and B_1 to be a permanently coincident point, the revolute joint between links 3 and 4, B. This vector equation can be written using complex numbers as:
\[ a_2e^{i\theta_{12}} + a_3e^{i\theta_{13}} = i\theta + s_{14} \]
This is a complex equation in three real unknowns, \( \theta_{12}, \theta_{13} \) and \( s_{14} \). If one of those variables (recall F = 1 for a slider-crank) is known the other two can be determined.

2. Kinematic Analysis

Vectors Allowed in a Loop Closure Equation
1. Body Vectors: They are the vectors that connect the two points on the same link. These vectors have a constant magnitude but the orientation may change.
2. Translational Joint Variable Vectors: They are the vectors between two links which are connected by a prismatic or cylinder in slot joint. These vectors are parallel to the relative sliding axis and have a variable magnitude. Direction may be variable as well.
3. Zero Vectors: They are the vectors connecting two permanently coincident points on two different links. They are not written!

2. Kinematic Analysis

Inverted Slider-Crank
This vector equation forces A_3 and A_2 to be a permanently coincident point, the revolute joint between links 2 and 3, A. This vector equation can be written using complex numbers as:
\[ a_2e^{i\theta_{12}} = a_3e^{i\theta_{13}} + s_{14}e^{i(\theta_{14}+\theta_4)} \]
This is a complex equation in three real unknowns, \( \theta_{12}, \theta_{13} \) and \( s_{14} \). If one of those variables (recall F = 1 for an inverted slider-crank) is known the other two can be determined.
2. Kinematic Analysis

Inverted Slider-Crank

\[ \begin{align*} 
A_0A_2 + A_0A_1 + A_2A_1 &= A_0B_9 + B_9B_8 + B_8A_1 + \ldots \\
0 &= a_1 + 0 + a_2e^{180} + s_3e^{180} \\
\text{Zero Vectors} \\
\text{Body Vectors} \\
\text{Translational Joint Variable Vector (a_1, b_1)}
\end{align*} \]

2. Kinematic Analysis

Multiloop Mechanisms

Euler’s polyhedron formula tells the number of independent loops as:

\[ L = j - \ell + 1 \quad \text{where} \quad j = \text{number of joints} \quad \ell = \text{number of loops} \quad L = \text{number of independent loops} \]

Next question: In which of the following five equations are independent?

\[ \begin{align*} 
A_0A + A_1B + BD &= A_0D + D_0D_6 (1) \\
A_0A + AC + CE &= A_0D' + D_0E_5 (2) \\
CE + ED &= CB + BD_6 (3) \\
A_0A + AC + CA &= 0 (4) \\
D_0D + CE + ED &= 0 (5)
\end{align*} \]

Problem: Is there a way to find out independent loop closure equations?

Yes, there is!

1. Disconnect gear pair(s) (if any) and write the gear relations.
2. Disconnect as many revolute joints as necessary to eliminate all loops. (However no link should be totally disconnected)
3. By connecting only one joint at a time (all others should be disconnected during this process) write the loop formed by connecting this joint.
2. Kinematic Analysis

Multiloop Mechanisms

\[ A_0 \dot{A} + \dot{\overrightarrow{AB}} + \overrightarrow{BD}_1 = \overrightarrow{AD}_0 + \overrightarrow{BD}_2 \]
\[ a_2 e^{\theta_{12}} + b_2 e^{i(\theta_{12}+\beta_2)} + a_3 e^{\theta_{13}} = a_1 + b_3 e^{i(\theta_{13}+\beta_3)} \]

Variables: \( \theta_{12}, \theta_{13}, \theta_{14}, \theta_{15} \) and \( \theta_{16} \)

Constraints: Two complex equations (i.e. four scalar equations)

\[ F = \lambda (\ell - j - 1) + \sum_{i=1}^{j} f_i = 3(6 - 7 - 1) + 7 = 1 \]
\[ V = 2L + F \]

2. Kinematic Analysis

Multiloop Mechanisms

1. Disconnect gear pair(s) and write the gear relation(s)
   \[ r_1 \theta_1 = -r_2 (\theta_2 - \beta) \]
2. Disconnect as many revolute joints as necessary to eliminate all loops.

Let's disconnect C. The number of joints to be disconnected is 3 as predicted by Euler's polyhedron formula: \( L = j - f + 1 = 9 - 7 + 1 = 3 \)

GP* is one of the joints the other two are \( C_{m} \) and \( C_{o} \) (please note that joint \( C_{m} \) is dependent on other two)

3. By connecting only one joint at a time (all others should be disconnected during this process write the loop formed by connecting this joint.

Reconnect \( C_m \), \( C_o \) and \( GP^{*} \) are disconnected)

\[ a \overrightarrow{BA} + b \overrightarrow{BD} + c \overrightarrow{BC} = a \overrightarrow{BA} + b \overrightarrow{BD} + c \overrightarrow{BC} \]

Equations: 5 (2 Complex LCE + Gear Relation)

\[ F = 1 \]
\[ V = 2L + F \]