2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

\[ s_{23}e^{i\theta_{12}} + a_3e^{i\left(\theta_{12}-\frac{\pi}{2}\right)} = a_1 + a_4e^{i\theta_{14}} \]

Re: \[ s_{23}\cos\theta_{12} + a_3\sin\theta_{12} = a_1 + a_4\cos\theta_{14} \]  
(1)

Im: \[ s_{23}\sin\theta_{12} - a_3\cos\theta_{12} = a_4\sin\theta_{14} \]  
(2)

1. Let \( \theta_{14} \) be the input

b. First solve \( \theta_{12} \) then \( s_{23} \)

\[ \sin\theta_{12}(1) - \cos\theta_{12}(2) \rightarrow a_3 = a_1\sin\theta_{12} + a_4(\cos\theta_{14}\sin\theta_{12} - \sin\theta_{14}\cos\theta_{12}) \]

\( (a_1 + a_4\cos\theta_{14})\sin\theta_{12} - a_4\sin\theta_{14}\cos\theta_{12} - a_3 = 0 \)

To solve this equation either half tangent or phase angle method may be utilized to obtain two closures.
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1. Let $\theta_{14}$ be the input

   b. First solve $\theta_{12}$ then $s_{23}$ (cont’ed)

   In general $s_{23}$ can be obtained either from (1) (except $\cos\theta_{12} = 0$) or from (2) (except $\sin\theta_{12} = 0$) however while performing full cycle analysis to avoid false singularities the following procedure works:

   \[
s_{23}\cos\theta_{12} = a_1 + a_4\cos\theta_{14} - a_3\sin\theta_{12} = x_{23}
   
   s_{23}\sin\theta_{12} = a_4\sin\theta_{14} + a_3\cos\theta_{12} = y_{23}
   \]

   Multiply $x_{23}$ by $\cos\theta_{12}$ and $y_{23}$ by $\sin\theta_{12}$

   \[
s_{23} = x_{23}\cos\theta_{12} + y_{23}\sin\theta_{12}
   \]

   which is free of singularities for all values of $\theta_{12}$. 

   $S_{23} < 0$
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\[ s_{23}e^{i\theta_{12}} + a_3e^{i\left(\theta_{12}-\frac{\pi}{2}\right)} = a_1 + a_4e^{i\theta_{14}} \]

Re: \[ s_{23}\cos\theta_{12} + a_3\sin\theta_{12} = a_1 + a_4\cos\theta_{14} \]  
Im: \[ s_{23}\sin\theta_{12} - a_3\cos\theta_{12} = a_4\sin\theta_{14} \]

2. Let \( s_{23} \) be the input

a. First solve \( \theta_{12} \) then \( \theta_{14} \)

\[ a_4\cos\theta_{14} = s_{23}\cos\theta_{12} + a_3\sin\theta_{12} - a_1 \]
\[ a_4\sin\theta_{14} = s_{23}\sin\theta_{12} - a_3\cos\theta_{12} \]

Sum of the squares of these

\[ a_4^2 = s_{23}^2 + a_1^2 + a_3^2 - 2a_1(s_{23}\cos\theta_{12} + a_3\sin\theta_{12}) \]

\[ s_{23}\cos\theta_{12} + a_3\sin\theta_{12} - \frac{s_{23}^2 + a_1^2 + a_3^2 - a_4^2}{2a_1} = 0 \]

Use half-tangent or phase angle method to solve and get two closures.
2. Kinematic Analysis

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Inverted Slider-Crank Mechanism

2. Let $s_{23}$ be the input

   a. First solve $\theta_{12}$ then $\theta_{14}$ (cont’ed)

   \[ a_4 \cos \theta_{14} = s_{23} \cos \theta_{12} + a_3 \sin \theta_{12} - a_1 = x_{14} \]
   \[ a_4 \sin \theta_{14} = s_{23} \sin \theta_{12} - a_3 \cos \theta_{12} = y_{14} \]
   \[ a_4, \theta_{14} = Pol(x_{14}, y_{14}) \]
2. Kinematic Analysis

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Inverted Slider-Crank Mechanism

2. Let $s_{23}$ be the input
2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Inverted Slider-Crank Mechanism

\[ s_{23} e^{i\theta_{12}} + a_3 e^{i\left(\theta_{12} - \frac{\pi}{2}\right)} = a_1 + a_4 e^{i\theta_{14}} \]

Re: \[ s_{23} \cos\theta_{12} + a_3 \sin\theta_{12} = a_1 + a_4 \cos\theta_{14} \quad (1) \]

Im: \[ s_{23} \sin\theta_{12} - a_3 \cos\theta_{12} = a_4 \sin\theta_{14} \quad (2) \]

2. Let \( s_{23} \) be the input
   b. First solve \( \theta_{14} \) then \( \theta_{12} \)
   Very similar to case a

3. Let \( \theta_{14} \) be the input
   Very similar to 1
2. Kinematic Analysis

3. Analytic–Closed Form Solution of Loop Closure Equations

Four-Bar Mechanism

\[ \overrightarrow{A_0A} + \overrightarrow{AB_3} = \overrightarrow{A_0B_0} + \overrightarrow{B_0B_4} \]

\[ a_2 e^{i\theta_{12}} + a_3 e^{i\theta_{13}} = a_1 + a_4 e^{i\theta_{14}} \]

Re: \[ a_2 \cos\theta_{12} + a_3 \cos\theta_{13} = a_1 + a_4 \cos\theta_{14} \] (1)

Im: \[ a_2 \sin\theta_{12} + a_3 \sin\theta_{13} = a_4 \sin\theta_{14} \] (2)

Let \( \theta_{12} \) be the input, solve \( \theta_{14} \) then \( \theta_{13} \)

\[ a_3 \cos\theta_{13} = a_1 + a_4 \cos\theta_{14} - a_2 \cos\theta_{12} \]

\[ a_3 \sin\theta_{13} = a_4 \sin\theta_{14} - a_2 \sin\theta_{12} \]

Square and add:

\[ 2a_1 a_4 \cos\theta_{14} - 2a_1 a_2 \cos\theta_{12} - 2a_2 a_4 (\cos\theta_{12} \cos\theta_{14} + \sin\theta_{12} \sin\theta_{14}) + a_1^2 + a_2^2 - a_3^2 + a_4^2 = 0 \]

Can be solved by half-tangent or phase angle method.

Then both \( \cos\theta_{13} \) and \( \sin\theta_{13} \) can be determined to find unique \( \theta_{13} \)
2. Kinematic Analysis

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Four-Bar Mechanism
2. Kinematic Analysis

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Take Home Message

1. The equation about the first variable to be solved is in one of the following forms:
   a. Quadratic (special case perfect square),
   b. Only sine (or cosine) of the angle is known,
   c. The equation is in a form $a\cos \theta + b\sin \theta + c = 0$.

   All these equations have two (very rarely double) solutions which determine the two closures of the mechanism.

2. The equation about the second variable always has a unique solution which depends on the closure selected in 1.
   a. Sine and cosine of the angle are both known,
   b. The unknown is linear in terms of knowns.

In general each loop has two closures so for a mechanism with $L$ loops the number of possible closures is $2^L$. 

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Returning back to Four-Bar Mechanism

\[2a_1a_4\cos \theta_{14} - 2a_1a_2\cos \theta_{12} - 2a_2a_4(\cos \theta_{12}\cos \theta_{14} + \sin \theta_{12}\sin \theta_{14}) + a_1^2 + a_2^2 - a_3^2 + a_4^2 = 0\]

Define:

\[K_1 = \frac{a_1}{a_2}, K_2 = \frac{a_1}{a_4}, K_3 = \frac{a_1^2 + a_2^2 - a_3^2 + a_4^2}{2a_2a_4}\]

and recall \(\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta\)

\[K_1 \cos \theta_{14} - K_2 \cos \theta_{12} + K_3 = \cos(\theta_{14} - \theta_{12})\]

Freudenstein’s Equation

Ferdinand Freudenstein (1926-2006)