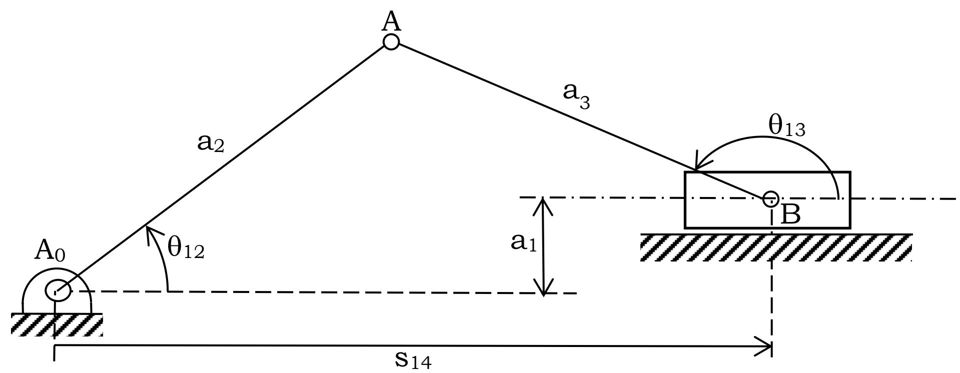
Exploration of Solution of Loop Closure Equations for RRRP (Slider-Crank and Inversions) Mechanisms

Based on Lecture Notes of Prof. Dr. M. Kemal ÖZGÖREN

1. Slider-Crank Mechanism



Loop closure equation in vector form: $\vec{A_0A_2} = \vec{A_0B} + \vec{BA_3}$

Loop closure equation in complex numbers: $a_2e^{i\theta_{12}} = s_{14} + ia_1 + a_3e^{i\theta_{13}}$

Real part of loop closure equation: $a_2\cos\theta_{12} = s_{14} + a_3\cos\theta_{13}$ (1.1)

Imaginary part of loop closure equation: $a_2\sin\theta_{12} = a_1 + a_3\sin\theta_{13}$ (1.2)

i. Let θ_{12} be input

a. First solve for s_{14} then θ_{13}:

Rearrange (1.1) and (1.2) as:

$$a_2\cos\theta_{12} - s_{14} = a_3\cos\theta_{13}$$

$$a_2\sin\theta_{12} - a_1 = a_3\sin\theta_{13}$$

Squaring the equations and adding them yields:

$$(a_2\cos\theta_{12} - s_{14})^2 + (a_2\sin\theta_{12} - a_1)^2 = a_3^2$$

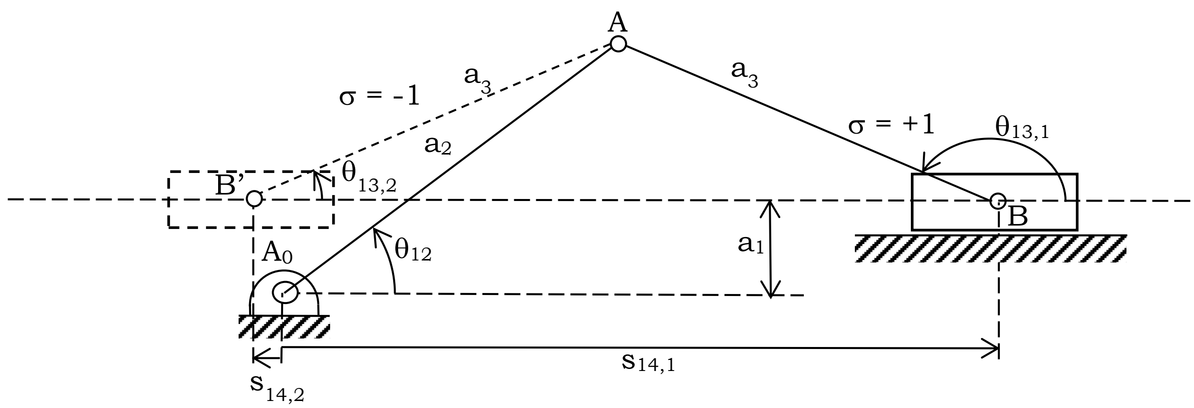
Expansion and refactoring yields:

$s_{14}^2 - (2a_2\cos\theta_{12})s_{14} + \left(a_2^2 + a_1^2 - a_3^2 - 2a_1a_2\sin\theta_{12}\right) = 0$ which is a quadratic in s_{14}, solution yields:

$$s_{14} = a_2 \cos\theta_{12} \pm \sqrt{(a_2 \cos\theta_{12})^2 - \left(a_2^2 + a_1^2 - a_3^2 - 2a_1 a_2 \sin\theta_{12}\right)}$$

s_{14} has a real solution only for $(a_2 \cos\theta_{12})^2 - \left(a_2^2 + a_1^2 - a_3^2 - 2a_1 a_2 \sin\theta_{12}\right) \geq 0$

Let $s_{14} = a_2 \cos\theta_{12} + \sigma\sqrt{(a_2 \cos\theta_{12})^2 - \left(a_2^2 + a_1^2 - a_3^2 - 2a_1 a_2 \sin\theta_{12}\right)}$ where $\sigma = \pm 1$ corresponds to the two different closures of the mechanism as:



After finding two possible solutions for s_{14}, the corresponding θ_{13} to each of the two possible s_{14} can be found by rearranging (1) and (2) as:

$$\cos\theta_{13} = \frac{a_2 \cos\theta_{12} - s_{14}}{a_3} = x_3$$

$$\sin\theta_{13} = \frac{a_2 \sin\theta_{12} - a_1}{a_3} = y_3$$

Therefore $\theta_{13} = a\tan 2(x_3, y_3)$

Please notice that for a selected closure (i.e. σ) there is one s_{14} and only one corresponding θ_{13}.

b. First solve for θ_{13} then s_{14}:

(1.2) does not contain s_{14} so rearranging (1.2) yields:

$$\sin\theta_{13} = \frac{a_2 \sin\theta_{12} - a_1}{a_3} = y_3$$

Recall $\sin^2\theta_{13} + \cos^2\theta_{13} = 1$ therefore $\cos\theta_{13} = \sigma\sqrt{1 - \sin^2\theta_{13}} = \sigma\sqrt{1 - y_3^2}$ where $\sigma = \pm 1$ corresponds to the two different closures of the mechanism and $\theta_{13} = a\tan 2(x_3, y_3)$.

Rearranging (1.1) as $s_{14} = a_2 \cos\theta_{12} - a_3 \cos\theta_{13}$ yields the corresponding s_{14} of the selected closure.

ii. Let s_{14} be the input

a. First solve θ_{12} then θ_{13}:

Rearrange (1.1) and (1.2) as:

$$a_2 \cos\theta_{12} - s_{14} = a_3 \cos\theta_{13}$$

$$a_2 \sin\theta_{12} - a_1 = a_3 \sin\theta_{13}$$

Squaring both and adding them yields:

$$\left(a_2 \cos\theta_{12} - s_{14}\right)^2 + \left(a_2 \sin\theta_{12} - a_1\right)^2 = a_3^2$$

Expansion and refactoring yields:

$$2a_2 s_{14} \cos\theta_{12} + 2a_1 a_2 \sin\theta_{12} - \left(a_2^2 + a_1^2 - a_3^2 + s_{14}^2\right) = 0$$ where θ_{12} is to be determined.

One way is to use *half-tangent* method as

Let $t_{12} = \tan\left(\frac{\theta_{12}}{2}\right)$ then $\cos\theta_{12} = \frac{1 - t_{12}^2}{1 + t_{12}^2}$ and $\sin\theta_{12} = \frac{2t_{12}}{1 + t_{12}^2}$.

Substitution yields

$$2a_2 s_{14} \frac{1 - t_{12}^2}{1 + t_{12}^2} + 2a_1 a_2 \frac{2t_{12}}{1 + t_{12}^2} - \left(a_2^2 + a_1^2 - a_3^2 + s_{14}^2\right) = 0$$

Rearranging yields:

$$\left(a_2^2 + a_1^2 - a_3^2 + s_{14}^2 - 2a_2 s_{14}\right) t_{12}^2 + 4a_1 a_2 t_{12} + \left(a_2^2 + a_1^2 - a_3^2 + s_{14}^2 + 2a_2 s_{14}\right) = 0$$

which is in the form $At_{12}^2 + Bt_{12} + C = 0$ where

$$A = a_2^2 + a_1^2 - a_3^2 + s_{14}^2 - 2a_2 s_{14}$$
$$B = 4a_1 a_2 t_{12}$$
$$C = a_2^2 + a_1^2 - a_3^2 + s_{14}^2 + 2a_2 s_{14}$$

$t_{12} = \dfrac{-B + \sigma\sqrt{B^2 + 4AC}}{2A}$ where $\sigma = \pm 1$ corresponds to the two different closures of the mechanism and $\theta_{12} = 2\tan^{-1}(t_{12})$. Please note that half tangent is single valued (therefore we can use inverse tangent function here)!

Other alternative to solve this equation is *phase angle* method. Rearranging yields:

$$s_{14}\cos\theta_{12} + a_1\sin\theta_{12} = \frac{a_2^{\,2} + a_1^{\,2} - a_3^{\,2} + s_{14}^{\,2}}{2a_2} \quad \text{or} \quad a\cos\theta_{12} + b\sin\theta_{12} = c$$

For $r > 0$ let $a = r\cos\phi$ and $b = r\sin\phi$ then $r = \sqrt{a^2 + b^2}$ and $\phi = a\tan 2(a, b)$.

Substitution yields

$$r\cos\varphi\cos\theta_{12} + r\sin\phi\sin\theta_{12} = c$$

which boils down to

$$r\cos(\varphi - \theta_{12}) = c$$

The solution yields $\theta_{12} = \varphi + \sigma\cos^{-1}\left(\dfrac{c}{r}\right)$ where $\sigma = \pm 1$ corresponds to the two different closures of the mechanism.

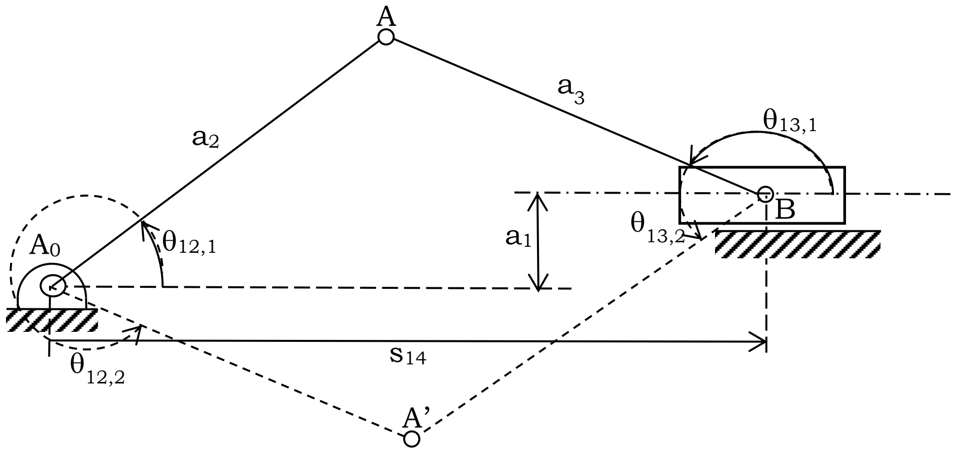
After finding two possible solutions for θ_{12}, for two possible closures of the mechanism, the corresponding θ_{13} to each of the two possible θ_{12} can be found by rearranging (1.1) and (1.2) as:

$$\cos\theta_{13} = \frac{a_2\cos\theta_{12} - s_{14}}{a_3} = x_3$$

$$\sin\theta_{13} = \frac{a_2\sin\theta_{12} - a_1}{a_3} = y_3$$

Therefore $\theta_{13} = a\tan 2(x_3, y_3)$

The two closures of the mechanism are:



b. First solve θ_{13} then θ_{12}

Squaring (1.1) and (1.2) and adding them eliminates θ_{12} as

$$a_2^2 = (s_{14} + a_3 \cos\theta_{13})^2 + (a_1 + a_3 \sin\theta_{13})^2$$

and simplification yields

$$s_{14}\cos\theta_{13} + a_1 \sin\theta_{13} = \frac{a_2^2 - s_{14}^2 - a_1^2 - a_3^2}{2a_3}$$

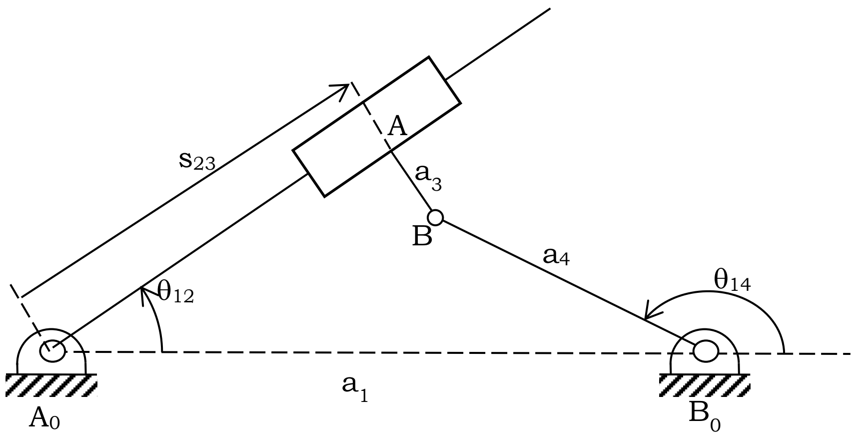
this equation can be solved either by half-tangent or phase angle method as presented before to find two θ_{13} values corresponding to two closures of the mechanism. Then

$$\cos\theta_{12} = \frac{a_3\cos\theta_{13} + s_{14}}{a_2} = x_2$$

$$\sin\theta_{12} = \frac{a_3\sin\theta_{13} + a_1}{a_2} = y_2$$

and $\theta_{12} = a\tan 2(x_2, y_2)$

2. Inverted Slider-Crank Mechanism



Loop closure equation in vector form: $\overrightarrow{A_0A}+\overrightarrow{AB_3}=\overrightarrow{A_0B_0}+\overrightarrow{B_0B_4}$

Loop closure equation in complex numbers: $a_2e^{i\theta_{12}}+a_3e^{i\left(\theta_{12}-\frac{\pi}{2}\right)}=a_1+a_4e^{i\theta_{14}}$

which boils down to $a_2e^{i\theta_{12}}-ia_3e^{i\theta_{12}}=a_1+a_4e^{i\theta_{14}}$

Real part of loop closure equation: $s_{23}\cos\theta_{12}+a_3\sin\theta_{12}=a_1+a_4\cos\theta_{14}$ (2.1)

Imaginary part of loop closure equation: $s_{23}\sin\theta_{12}-a_3\cos\theta_{12}=a_4\sin\theta_{14}$ (2.2)

i. Let θ_{14} be input

a. First solve for s_{23} then θ_{12}:

Square (2.1) and (2.2) and add:

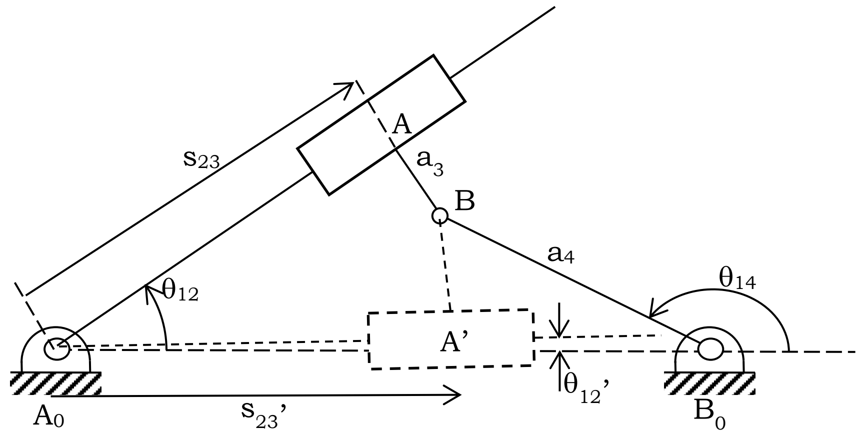
$$s_{23}{}^2+a_3{}^2=a_1{}^2+2a_1a_4\cos\theta_{14}+a_4{}^2$$

yielding $s_{23}=\sigma\sqrt{a_1{}^2-a_3{}^2+a_4{}^2+2a_1a_4\cos\theta_{14}}$ where $\sigma=\pm1$ corresponds to the two different closures of the mechanism.

To find θ_{12} let $\cos\theta_{12}=x_2$ and $\sin\theta_{12}=y_2$, substituting (1) and (2) into matrix form:

$$\begin{bmatrix} s_{23} & -a_3 \\ a_3 & s_{23} \end{bmatrix}\begin{Bmatrix} y_3 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} a_4\sin\theta_{14} \\ a_1+a_4\cos\theta_{14} \end{Bmatrix}$$ from which $\theta_{12}=a\tan 2\left(x_2,y_2\right)$

The two closures of the mechanism are



b. First solve for θ_{12} then s_{23}

Using equations (2.1) and (2.2), $(2.1)\sin\theta_{12} - (2.2)\cos\theta_{12}$ yields

$$a_3 = a_1 \sin\theta_{12} + a_4(\cos\theta_{14}\sin\theta_{12} - \sin\theta_{14}\cos\theta_{12})$$

rearranging yields $(a_1 + a_4\cos\theta_{14})\sin\theta_{12} - (a_4\sin\theta_{14})\cos\theta_{12} - a_3 = 0$ which can either be solved by half-tangent substitution or phase angle method as presented before where the two solutions correspond to two closures of the mechanism.

To determine s_{23} corresponding to the selected closure (i.e. θ_{12}) rearrange (2.1) and (2.2) as:

$$s_{23}\cos\theta_{12} = a_1 + a_4\cos\theta_{14} - a_3\sin\theta_{12} = x_3$$

$$s_{23}\sin\theta_{12} = a_4\sin\theta_{14} + a_3\cos\theta_{12} = y_3$$

Multiply x_3 by $\cos\theta_{12}$ and y_3 by $\sin\theta_{12}$ to obtain:

$$s_{23} = x_3\cos\theta_{12} + y_3\sin\theta_{12}$$

Please note that this final equation will be free of singularities of $\sin\theta_{12}$ or $\cos\theta_{12}$ being zero during full cycle analysis which are false singularities.

ii. Let θ_{12} be input

a. First solve for θ_{14} then s_{23}:

Using equations (2.1) and (2.2), $(2.1)\sin\theta_{12} - (2.2)\cos\theta_{12}$ yields

$$a_3 = a_1 \sin\theta_{12} + a_4(\cos\theta_{14}\sin\theta_{12} - \sin\theta_{14}\cos\theta_{12})$$

rearranging yields $(a_4\cos\theta_{12})\sin\theta_{14} - (a_4\sin\theta_{12})\cos\theta_{14} + a_3 - a_1\sin\theta_{12} = 0$ which can either be solved by half-tangent substitution or phase angle method as presented before where the two solutions correspond to two closures of the mechanism. Corresponding s_{23} can be found as presented in section 2.i.b to avoid mentioned false singularities.

b. First solve for s_{23} then θ_{14}:

Sum of squares of (2.1) and (2.2) yields:

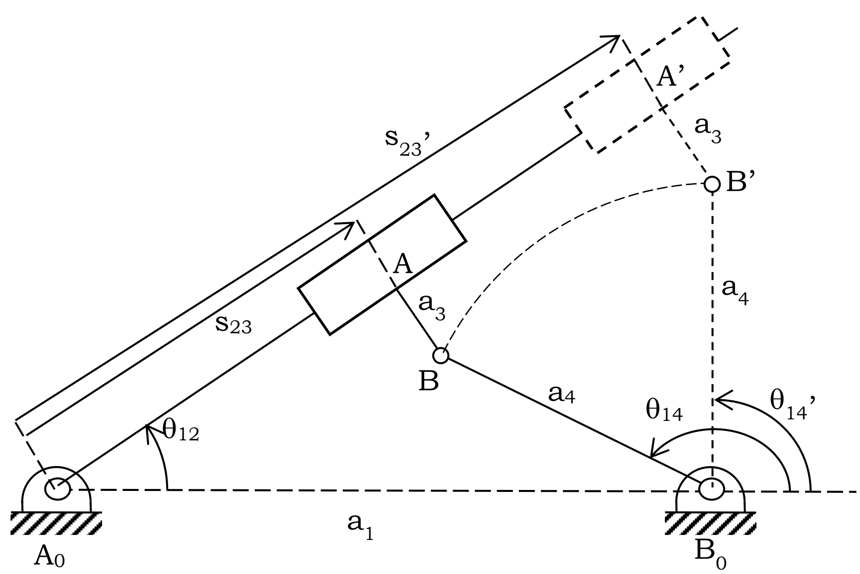
$s_{23}^2 - (2a_1\cos\theta_{12})s_{23} + \left(a_1^2 + a_3^2 - a_4^2 - 2a_1a_3\sin\theta_{12}\right) = 0$ which is a quadratic in s_{23} whose two solutions correspond to two closures of the mechanism. To determine the corresponding θ_{14} to the selected closure (2.1) and (2.2) can be rearranged as:

$$\cos\theta_{14} = \frac{s_{23}\cos\theta_{12} + a_3\sin\theta_{12} - a_1}{a_4} = x_4$$

$$\sin\theta_{14} = \frac{s_{23}\sin\theta_{12} - a_3\cos\theta_{12}}{a_4} = y_4$$

so $\theta_{14} = a\tan 2(x_4, y_4)$

The two closures are:



iii. Let s_{23} be input (can be a piston or linear actuator)

a. First solve for θ_{12} then θ_{14}:

Rearrange (2.1) and (2.2) as:

$$s_{23}\cos\theta_{12} + a_3\sin\theta_{12} - a_1 = a_4\cos\theta_{14}$$

$$s_{23}\sin\theta_{12} - a_3\cos\theta_{12} = a_4\sin\theta_{14}$$

Square these equations and add to obtain

$\left(s_{23}\right)\cos\theta_{12} + \left(a_3\right)\sin\theta_{12} - \left(\dfrac{s_{23}^{\ 2} + a_3^{\ 2} + a_1^{\ 2} - a_4^{\ 2}}{2a_1}\right) = 0$ which can either be solved by half-tangent substitution or phase angle method as presented before where the two solutions correspond to two closures of the mechanism. To determine the corresponding θ_{14} to the selected closure (2.1) and (2.2) can be rearranged as:

$$\cos\theta_{14} = \frac{s_{23}\cos\theta_{12} + a_3\sin\theta_{12} - a_1}{a_4} = x_4$$

$$\sin\theta_{14} = \frac{s_{23}\sin\theta_{12} - a_3\cos\theta_{12}}{a_4} = y_4$$

so $\theta_{14} = a\tan 2\left(x_4, y_4\right)$

The two closures are: