EE 503 Quiz 1

Nov. 9, 2020

Duration: 25 minutes

Problem (25 pts)

The norm function of a real-valued N-dimensional vector space is given as $\|x\|_W = \sqrt{x^T W x}$.

a) What are the conditions (if any) on matrix W for the norm definition to be valid?
b) Assume x vector is 2×1 vector with entries x_1 and x_2. The units of x_1 and x_2 are centimeters and meters, respectively. Can you suggest a matrix W such that $\|x\|_W$ has the unit of meters?
c) Find the expression for x that minimizes $J(x) = \|Ax - b\|_W^2$ where A is a full column rank matrix.

Answer:

+5 a) $\underline{W} > 0$, otherwise $\left(\|x\|_W \geq 0 \ \forall x\right)$ or $\left(\|x\|_W = 0 \leftrightarrow x = 0\right)$ won't be satisfied.

Other norm axioms (scaling and Δ-inequality) are automatically satisfied, since $\|x\|_W = \|\hat{x}\|$, where $\hat{x} = W^{1/2} x$ — standard Euclidean norm.

(Note that: $W^{1/2}$ exists if $W > 0$)

+5 b) $\underline{x} = \begin{bmatrix} \alpha\,(cm) \\ \beta\,(m) \end{bmatrix} \rightarrow x^T W x = \alpha^2 w_1 + \beta^2 w_2$; $\begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}$; cm^2, m^2 ← unit conversion factors

Set $w_1 = 10^{-4} \left(m^2/cm^2\right)$, $w_2 = 1\ m^2/m^2 \rightarrow x^T W x$ has the unit of m^2 and $\sqrt{x^T W x}$ has the unit of meters.

+15 c) $J(x) = \|Ax - b\|_W^2 = (Ax - b)^T W (Ax - b) = x^T A^T W A x - b^T W A x - x^T A^T W b + \|b\|^2$

$\nabla_x J(x) = \left(A^T W A + A^T W^T A\right) x - A^T W b - A^T W^T b = 0$

$x = \left(A^T W A + A^T W^T A\right)^{-1} \left(A^T W + A^T W^T\right) b$

$= \left(A^T (W + W^T) A\right)^{-1} A^T (W + W^T) b$.

Scanned with CamScanner