EE 503 Quiz 2

Nov. 30, 2020

Duration: 30 minutes

Problem (25 pts)

The random process $x(t)$ is a zero mean Gaussian process with the auto-correlation function $r_x(t_1, t_2) = 4^{-|t_1 - t_2|}$.

a)
  i. Write the mathematical expression for the density of $x(10)$.
  ii. Write the mathematical expression for the joint density of $x(10)$ and $x(11)$.
  iii. Is $x(t)$ a stationary process? If yes, state its type.

b) The process $y(t)$ is defined as $y(t) \triangleq x(t) + x(t-1)$.
  i. Find the first two moment description of $y(t)$.
  ii. Find the first two order joint pdf description of $y(t)$.
  iii. Is $y(t)$ a Gaussian process? Explain.
  iv. Is $y(t)$ a stationary process? If yes, state its type. Explain your reasoning.

c) The process $z(t)$ is defined as $z(t) \triangleq x(t) + x\left(\frac{t}{2}\right)$.
  i. Find the first two moment description of $z(t)$.
  ii. Find the first two order joint pdf description of $z(t)$.
  iii. Is $z(t)$ a Gaussian process? Explain.
  iv. Is $z(t)$ a stationary process? If yes, state its type. Explain your reasoning.

Answer:

Please see the next page.

EE 503

Quiz #2

$x(t)$: Gaussian process, $\mu_x(t)=0$, $r_x(t_1,t_2)=4^{-|t_1-t_2|}$

a) $z = x(10) \sim N(0, r_x(10,10))$ ← $f_z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-\frac{(z-\mu_z)}{2\sigma_z^2}}$ ← $N(\mu_z, \sigma_z^2)$

$1 = \sigma_z^2$

i) $\mu_z = 0$, $\sigma_z^2 = 1$

$z = x(10)$, $w = x(11)$ } $v = \begin{bmatrix} z \\ w \end{bmatrix}$ ← $f_{z,w}(z,w) = \frac{1}{2\pi |C_v|^{1/2}} e^{-\frac{1}{2}[z\ w] C_v^{-1} \begin{bmatrix} z \\ w \end{bmatrix}}$ ← $N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, C_v\right)$

ii) $\mu_v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C_v = E\left\{\begin{bmatrix} z \\ w \end{bmatrix}[z\ w]\right\} = \begin{bmatrix} r_x(10,10) & r_x(10,11) \\ r_x(11,10) & r_x(11,11) \end{bmatrix}$

$\begin{bmatrix} 1 & 1/4 \\ 1/4 & 1 \end{bmatrix}$

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iii) Yes! $x(t)$ is stationary in WSS since $\mu_x(t) = \text{constant}$, $r_x(t_1,t_2) = \text{func}(t_1-t_2)$

Since $x(t)$ is Gaussian process and WSS → $x(t)$ is stationary in N^{th} joint pdf for all N. (SSS)

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b) $y(t) = x(t) + x(t-1)$

i) $\mu_y(t) = \mu_x(t) + \mu_x(t-1) = 0$ (each term $\to 0$)

$r_y(t_1,t_2) = E\{[x(t_1)+x(t_1-1)][x(t_2)+x(t_2-1)]\}$

$= r_x(t_1,t_2) + r_x(t_1-1,t_2) + r_x(t_1,t_2-1) + r_x(t_1-1,t_2-1)$

$= 4^{-|t_1-t_2|} + 4^{-|t_1-t_2-1|} + 4^{-|t_1-t_2+1|} + 4^{-|t_1-t_2|}$

$= 2\cdot 4^{-|z|} + 4^{|z-1|} + 4^{-|z+1|}$ ← $r_y(z)$ $(z \triangleq t_1-t_2)$

ii) $y(t) \sim N(0, r_y(z))$, $z=0$ ← $N(0, 2+\frac{1}{2})$

$(\Delta = t_1 - t_2)$

$\begin{bmatrix} y(t_1) \\ y(t_2) \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_y(t_1,t_1) & r_y(t_1,t_2) \\ r_y(t_2,t_1) & r_y(t_2,t_2) \end{bmatrix}\right)$ ← $\begin{bmatrix} r_y(0) & r_y(+\Delta) \\ r_y(-\Delta) & r_y(0) \end{bmatrix}$

iii) Yes! y(t) is a Gaussian process; since linear combination of jointly Gaussian r.v.'s (samples of Gaussian process) is jointly Gaussian.

iv) Yes! $\mu_y(t) = \text{constant} \leftarrow 0$, $r_y(t_1, t_2) = \text{func}(\underbrace{t_1 - t_2}_{\Delta})$ } Hence, y(t) is WSS

Since y(t) is Gaussian and WSS, y(t) is SSS.

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c) $z(t) = x(t) + x\left(\frac{t}{2}\right)$

i) $E\{z(t)\} = \mu_x(t) + \mu_x\left(\frac{t}{2}\right) = 0$

$$r_z(t_1, t_2) = E\left\{\left[x(t_1) + x\left(\frac{t_1}{2}\right)\right]\left(x(t_2) + x\left(\frac{t_2}{2}\right)\right)\right\}$$

$$= r_x(t_1, t_2) + r_x\left(\frac{t_1}{2}, t_2\right) + r_x\left(t_1, \frac{t_2}{2}\right) + r_x\left(\frac{t_1}{2}, \frac{t_2}{2}\right)$$

$$= 4^{-|t_1 - t_2|} + 4^{-\left|\frac{t_1}{2} - t_2\right|} + 4^{-\left|t_1 - \frac{t_2}{2}\right|} + 4^{-\left|\frac{t_1}{2} - \frac{t_2}{2}\right|}$$

$$= 4^{-|t_1 - t_2|} + 2^{-|t_1 - 2t_2|} + 2^{-|2t_1 - t_2|} + 2^{-|t_1 - t_2|}$$

ii) $z(t_1) \sim N(0, r_z(t_1, t_1)) \leftarrow N(0, 2 + 2 \cdot 2^{-|t_1|})$ ← depends on t_1!

$$\begin{bmatrix} z(t_1) \\ z(t_2) \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_z(t_1, t_1) & r_z(t_1, t_2) \\ r_z(t_2, t_1) & r_z(t_2, t_2) \end{bmatrix}\right)$$ ← depends on both t_1 and t_2!

iii) Yes! y(t) is a Gaussian process, since the explanation in part b-iii is a valid in this case.

iv) No! $\mu_z(t) = 0$, $r_z(t_1, t_2) \neq \text{func}(t_1 - t_2)$ } → Not WSS