EE 503 Quiz 3

Jan. 15, 2021

Duration: 25 minutes

Problem (25 pts)

The random processes $s[n]$ and $v[n]$ are zero mean, uncorrelated processes with the auto-correlation sequences $r_s[k] = \sigma_s^2 \alpha^{|k|}$ and $r_v[k] = \sigma_v^2 \beta^{|k|}$, respectively. Assume that we observe the superposition of two processes which is denoted as $x[n]$:

$$x[n] = s[n] + v[n].$$

a) Write the Wiener-Hopf equations for the LMMSE estimation of $s[n]$ given the observations $x[n]$ and x$[n-1]$, $\hat{s}_a[n] = w_0 x[n] + w_1 x[n-1]$. (No need to solve for w_0 and w_1.)

b) Write the Wiener-Hopf equations for the LMMSE estimation of $s[n+1]$ given the observations $x[n]$ and x$[n-1]$, $\hat{s}_b[n+1] = z_0 x[n] + z_1[n-1]$. (No need to solve for z_0 and z_1.)

c) Discuss the optimality of the estimator $\alpha \hat{s}_a[n]$, where $\hat{s}_a[n]$ is the estimator in part-a, for the solution of the estimation problem in part-b.

Home Study: (Do not answer part-d in Quiz time!)

d) Discuss the optimality of the estimator $\hat{v}[n] = x[n] - \hat{s}_a[n]$ for the estimation of $v[n]$ given the observations $x[n]$ and x$[n-1]$. Here $\hat{s}_a[n]$ is the estimator in part-a. Is the estimator $\hat{v}[n] = x[n] - \hat{s}_a[n]$ optimal in any sense?

EE 503 — Quiz 3 Solution

$s[n]: \; r_s[k] = \sigma_s^2 \alpha^{|k|}$
$v[n]: \; r_v[k] = \sigma_v^2 \beta^{|k|}$

$\underbrace{x[n]}_{\text{observations}} = \underbrace{s[n]}_{\text{desired process}} + \underbrace{v[n]}_{\text{measurement noise}}$, ($s[n]$ and $v[n]$ uncorrelated)

a) $\underline{x} = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$, $d = s[n] \rightarrow \underline{\underline{R}}_x \underline{w} = r_{d\underline{x}}$

$$\left(\begin{aligned} r_x[k] &= r_s[k] + r_v[k] \\ &= \sigma_s^2 \alpha^{|k|} + \sigma_v^2 \beta^{|k|} \\ r_{dx}[k] &= r_s[k] \end{aligned} \right)$$

$$\underbrace{\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(-1) & r_x(0) \end{bmatrix}}_{E\{\underline{x}\underline{x}^T\}} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \underbrace{\begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix}}_{E\{d\underline{x}\}} \begin{array}{l} \leftarrow r_s[0] \\ \leftarrow r_s[1] \end{array}$$

$$\begin{bmatrix} \sigma_s^2 + \sigma_v^2 & \sigma_s^2\alpha + \sigma_v^2\beta \\ \sigma_s^2\alpha + \sigma_v^2\beta & \sigma_s^2 + \sigma_v^2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} \sigma_s^2 \\ \sigma_s^2\alpha \end{bmatrix}$$

b) $\underline{x} = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$, $d = s[n+1] \rightarrow$

$$\begin{bmatrix} r_x(0) & r_x(1) \\ r_x(-1) & r_x(0) \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ r_{dx}(1) \end{bmatrix} \begin{array}{l} \leftarrow r_s[1] \\ \leftarrow r_s[2] \end{array}$$

$$\left(\begin{aligned} r_{dx}(k) &= E\{d\,x[n-k]\} \\ &= E\{s[n+1](s[n-k] + v[n-k])\} \\ &= r_s[k+1] \end{aligned} \right)$$

$$\begin{bmatrix} \sigma_s^2 + \sigma_v^2 & \sigma_s^2\alpha + \sigma_v^2\beta \\ \sigma_s^2\alpha + \sigma_v^2\beta & \sigma_s^2 + \sigma_v^2 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} \alpha\sigma_s^2 \\ \alpha^2\sigma_s^2 \end{bmatrix}$$

c) Note that $\underline{\underline{R}}_x \underline{w} = \sigma_s^2 \begin{bmatrix} 1 \\ \alpha \end{bmatrix}$ and $\underline{\underline{R}}_x \underline{z} = \sigma_s^2 \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \alpha$ are the equation systems for the optimal coefficients; since LHS of both equations are same and RHS are identical apart from scaling with α, we have

$$\underline{w} = \underline{\underline{R}}_x^{-1}\begin{bmatrix}1\\ \alpha\end{bmatrix}\sigma_s^2 \quad \text{and} \quad \underline{z} = \underbrace{\left(\underline{\underline{R}}_x^{-1}\begin{bmatrix}1\\ \alpha\end{bmatrix}\sigma_s^2\right)}_{\underline{w}}\alpha$$

and therefore $\underline{z} = \underline{w}.\alpha$

coefficients for the solution of part-a

coef. for the sol. of part-b.

Then, we have

$$\hat{s}_a[n] = \underline{w}^T\underline{x} \quad \text{and} \quad \hat{s}_b[n+1] = \underline{z}^T x$$

Wiener filter soln. for part-a

Wiener filter soln. for part-b

and $\hat{s}_b[n+1] = \alpha\hat{s}_a[n+1]$.

This result is not surprising since $r_s[k] = \sigma_s^2\alpha^{|k|}$ information states that $s[n] = \alpha s[n-1] + (\text{white-noise})[n]$ is the synthesis equation for the process $s[n]$. Stated differently, we have $s[n+1] = \alpha s[n] + (\text{white-noise}[n+1])$. From part-a we know that $E\left\{\left(s[n]-\hat{s}_a[n]\right)\begin{bmatrix}x(n)\\ x(n-1)\end{bmatrix}\right\} = \begin{bmatrix}0\\0\end{bmatrix}$ (orthogonality condition). is satisfied. The orthogonality cond. for part-b is

$$E\left\{\left(s[n+1]-\hat{s}_b[n]\right)\begin{bmatrix}x(n)\\ x(n-1)\end{bmatrix}\right\} = \begin{bmatrix}0\\0\end{bmatrix}$$

Let's substitute $\hat{s}_b[n] = \alpha \hat{s}_a[n]$ and $s[n+1] = \alpha s[n] + (\text{white-noise}[n+1])$ into

the orth. cond. for part-b and get

$$E\left\{ \Big(\underbrace{\alpha s[n] + (\text{white-noise}[n+1])}_{s[n+1]} - \underbrace{\alpha \hat{s}_a[n]}_{\hat{s}_b[n]} \Big) \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} \right\} =$$

$$\cdots = \alpha \underbrace{E\left\{ (s[n] - \hat{s}_a[n]) \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} \right\}}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \underbrace{E\left\{ \begin{pmatrix} \text{white} \\ \text{noise} \\ [n+1] \end{pmatrix} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} \right\}}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The process noise to generate $s[n]$ with sample index $n+1$ is ~~independent~~ uncorrelated with earlier samples $s[n]$ and observation noise $v[n]$.

So, $\alpha \hat{s}_a[n]$ ~~also~~ satisfies the orthogonality condition for part-b. (This part is related to Property-4 discussed in Lec. 36a of LMMSE estimators.)