METU, Ankara
Fall 2020-21

EE 503
Quiz 4

Conducted in Final Exam week
(Duration: 15 minutes)

1. (Quiz #4, 15 minutes) The random vector $\mathbf{x}$ has the following correlation matrix:

$$\mathbf{R}_x = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

a) Find the basis vectors for the KL transform associated with the given auto-correlation matrix.

b) Assume that the random vector $\mathbf{x}$ is approximated by projecting the vector $\mathbf{x}$ to a 1-dimensional subspace. Find the sub-space with the minimum mean square approximation error $E\{||\mathbf{x}-\hat{\mathbf{x}}||^2\}$ where $\hat{\mathbf{x}} = \mathbf{P_u}\mathbf{x}$ and $\mathbf{P_u}$ is the orthogonal projector to $\mathbf{u}$.

c) What is the total MSE $E\{||\mathbf{x}-\hat{\mathbf{x}}||^2\}$ of the approximation with the optimal $\mathbf{u}$?

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Problem:

$$\underline{R}_x = \left[\begin{array}{cc|cc} 4 & 1 & \multicolumn{2}{c}{\underline{0}} \\ 1 & 4 & & \\ \hline \multicolumn{2}{c|}{\underline{0}} & 2 & 1 \\ & & 1 & 2 \end{array}\right]$$

a) $\underline{R}_x \underline{e}_k = \lambda_k e_k \rightarrow \underline{R}_x \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} = \lambda_k \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda_k \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow$

+15

$\rightarrow \underline{R}_x \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix} = \lambda_k' \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \lambda_k' \begin{bmatrix} c \\ d \end{bmatrix} \rightarrow$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_k = 5$

$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_k = 3$

$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_k' = 3$

$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \lambda_k' = 1$

Hence, $\begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}$ are the eigenvectors of R_x

$\rightarrow \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\lambda^{(1)}_{R_x} = 5 \quad \lambda^{(2)}_{R_x} = 3 \quad \lambda^{(3)}_{R_x} = 3 \quad \lambda^{(4)}_{R_x} = 1.$

The eigenvector set form an orthogonal basis for R^4!

b) From KL transformation result the optimal 1 sub-space should be spanned by the eigenvector with largest eigenvalue.

Then $\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $\underline{\underline{P}}_v = \frac{\underline{v}\underline{v}^T}{\|v\|^2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot [1\ 1\ 0\ 0] = \frac{1}{2} \left[\begin{array}{c|c} \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \underline{\underline{0}} \\ \hline \underline{\underline{0}} & \underline{\underline{0}} \end{array}\right]_{4\times4}$

(+5)

c) Assume $\underline{x}$ is expressed in terms of the eigenvectors of $\underline{\underline{R}}_x$, that is $\underline{x} = \langle x, \underline{e}_1^n \rangle \underline{e}_1^n + \langle \underline{x}, \underline{e}_2^n \rangle e_2^n + \langle x, \underline{e}_3^n \rangle e_3^n + \langle x, \underline{e}_4^n \rangle \underline{e}_4^n$

(+5)

where $\underline{e}_k^n = \frac{\underline{e}_k}{\|e_k\|}$, that is the normalized (unit norm) version of $\underline{e}_k$. Then $\underline{x} - \hat{\underline{x}} = \langle \underline{x}, \underline{e}_2^n \rangle \underline{e}_2^n + \langle \underline{x}, \underline{e}_3^n \rangle \underline{e}_3^n + \langle x, e_4^n \rangle \underline{e}_4^n$

and $\|\underline{x} - \hat{\underline{x}}\|^2 = \sum_{k=2}^{4} |\langle \underline{x}, \underline{e}_k^n \rangle|^2 = \sum_{k=2}^{4} \underline{e}_k^{n} \underline{x}\,\underline{x}^T \underline{e}_k^n$.

Finally,

$$E\{\|\underline{x} - \hat{\underline{x}}\|^2\} = \sum_{k=2}^{4} \underline{e}_k^n \underline{\underline{R}}_x \underline{e}_k^n = \sum_{k=2}^{4} \lambda_{R_x}^{(k)} \underbrace{\|\underline{e}_k^n\|^2}_{1} = \sum_{k=2}^{4} \lambda_{R_x}^{(k)} = 3+3+1$$

$$= 7$$