EE 503
Homework #3
(Due: Feb. 2, 2021 - 17:30)

We examine a channel equalization problem in this assignment. Related discussions can be found in the digital communications textbooks and in [pages 530-534, Hayes] (also see [pages 369-371, Hayes).

Problem:

In the following set-up $h_{ch}[n]$ shows the impulse response of a causal FIR channel. We would like to design $h_{eq}[n]$, which is the equalizer filter, to compensate the effect of the channel. The random process $w[n]$ is zero mean, white Gaussian noise (WGN) process with variance σ_w^2. The process $s[n]$ is the information carrying process that we are interested in. The process $s[n]$ is uncorrelated with $w[n]$.

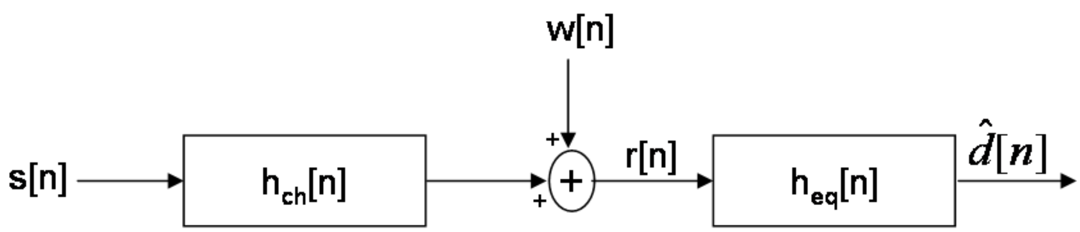

Figure: Block diagram for the channel equalization set-up

a) Consider $H_{ch}(z) = h_{ch}[0] + h_{ch}[1]z^{-1} + ... + h_{ch}[L]z^{-L}$ and $d[n] = s[n-\Delta]$ where Δ is a non-negative integer. Write the expressions for the auto-correlation sequence $r_r[k]$ and the cross-correlation sequence $r_{dr}[k]$.

b) Take $H_{ch}(z) = \frac{14}{5} + \frac{2}{5}z^{-1} + \frac{4}{5}z^{-2} + 2z^{-3} + \frac{1}{5}z^{-4} - \frac{1}{5}z^{-5}$ and design FIR equalizers of order P, i.e. $H_{eq}(z) = h_{eq}[0] + h_{eq}[1]z^{-1} + ... + h_{eq}[P]z^{-P}$, such that $E\left\{\left(d[n] - \hat{d}[n]\right)^2\right\}$ is minimized.

   Note that if it is possible to reduce the error of $E\left\{\left(d[n] - \hat{d}[n]\right)^2\right\}$ to zero (ideal equalizer), then the cascade of $h_{ch}[n]$ and $h_{eq}[n]$ becomes $\delta[n-\Delta]$; hence after the equalization the cascade channel becomes the delay channel with colored additive noise.

   For the design of the equalization filters assume that $r_s[k] = \delta[k]$ and $r_w[k] = \frac{1}{SNR}\delta[k] = \sigma_w^2\delta[k]$.

   Here $SNR = \frac{r_s[0]}{r_w[0]} = \frac{\sigma_s^2}{\sigma_w^2}$, where $\sigma_s^2 = r_s[0] = 1$, shows the ratio of signal and noise powers.

Submit the following and related MATLAB codes:

1. Plot of $conv(h_{ch}[n], h_{eq}[n])$ at SNR = 10 dB, $\Delta = 2$ for $P = \{3,5,7,11\}$.
2. Plot of $conv(h_{ch}[n], h_{eq}[n])$ at SNR = 10 dB, $\Delta = 4$ for $P = \{3,5,7,11\}$.
3. Using Matlab calculate the error of $J_{\min} = E\left\{\left(d[n] - \hat{d}[n]\right)^2\right\}$ of the optimum equalizer and fill in the following table for each pair of P and Δ at SNR = 10 dB.

Jmin	Δ=3	4	7	8	11	Δ=12
P=3						
4						
7						
8						
10						
11						
P=12						

Achieved Minimum Error for Various Configurations for SNR = 10 dB

4. Calculate the error of $J_{\min} = E\left\{\left(d[n] - \hat{d}[n]\right)^2\right\}$ of the optimum equalizer and fill in the following table for each pair of P and Δ at SNR = 0 dB.

Jmin	Δ=3	4	7	8	11	Δ=12
P=3						
4						
7						
8						
10						
11						
P=12						

Achieved Minimum Error for Various Configurations for SNR = 0 dB

Notes:

- For parts 3 and 4, comment on the effect of SNR. Is the performance significantly affected by SNR?
- State the values for P and Δ for which the equalizer is all zero filter. These cases should appear in the tables with J_{min}=1. Explain whether this makes sense or not.

c) In this part, we numerically examine the success the effect of equalization on a communication scheme. This part shows how the equalizer is used in a practice.

For this part, assume that $s[n]$ is equally likely to be *1* or *-1* for every n and $s[n]$ is i.i.d. distributed. (Convince yourself that this is compatible with $r_s[k] = \delta[k]$.)

1. Generate $s[n]$ for $n = \{0,1,\ldots,99\}$ and implement the filtering scheme shown in the figure with $H_{ch}(z) = \frac{14}{5} + \frac{2}{5}z^{-1} + \frac{4}{5}z^{-2} + 2z^{-3} + \frac{1}{5}z^{-4} - \frac{1}{5}z^{-5}$, for $\Delta = 2$, $P = 5$ and SNR = 10 dB.

Present $s[n-\Delta]$ and $\hat{d}[n]$ using subplot command of Matlab. Compare $s[n-\Delta]$ and $\hat{d}[n]$. Do you think channel equalization is successful?

To get a numerical value for the success of equalization, empirically estimate the probability of error by counting how many times $\text{sign}(\hat{d}[n]) \neq s[n-\Delta]$. Use the following sign function definition:

$$sign(x) = \begin{cases} 1 & x > 0 \\ \pm 1\left(\text{with prob. } \frac{1}{2}\right) & x = 0 \\ -1 & x < 0 \end{cases}$$

2. Repeat the earlier part for $s[n]$ for $n = \{0,1,\ldots,99999\}$ and provide your empirical estimate for the probability of error (as in earlier part) for each pair of P and Δ at SNR values of 10 and 0 dB.

P{Error}	Δ=3	4	7	8	11	Δ=12
P=3						
4						
7						
8						
10						
11						
P=12						

Probability of Error Estimate for SNR = 10 dB

P{Error}	Δ=3	4	7	8	11	Δ=12
P=3						
4						
7						
8						
10						
11						
P=12						

Probability of Error Estimate for SNR = 0 dB

3. Quantitatively compare the approach in this homework assignment with the one in HW-2. Note that in HW-2, the equalization is done by taking into account the observation system (which is a system convolving the input with a low-pass sequence denoted as *g[n]*); but not the statistical characteristics of noise and the desired signal. The pre-whitening operation and its weight in HW-2 is suggested as an ad-hoc solution when the performance turned out to be very poor in the presence of observation noise. Compare the approach with the pre-whitening operation with the one in this homework.