Q0. It is crucial that you review the basics of linear algebra. You should be able to solve the questions in the problem sets on `www.eee.metu.edu.tr/~tuna/ee501/`

Q1. Find the state space representations for the following systems. That is, choose your state variables and put your system into

$$\left\{\begin{array}{lcl} \dot{x} & = & f(x,\,u) \\ y & = & g(x,\,u) \end{array}\right\} \text{ if NL, } \left\{\begin{array}{lcl} \dot{x} & = & A(t)x + B(t)u \\ y & = & C(t)x + D(t)u \end{array}\right\} \text{ if LTV, } \left\{\begin{array}{lcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array}\right\} \text{ if LTI.}$$

(a) $\ddot{y} + 3\dot{y} + y = \ddot{u} + u$

(b) $\ddot{y} + \dot{y} + y = u^2 + \sin(u)$

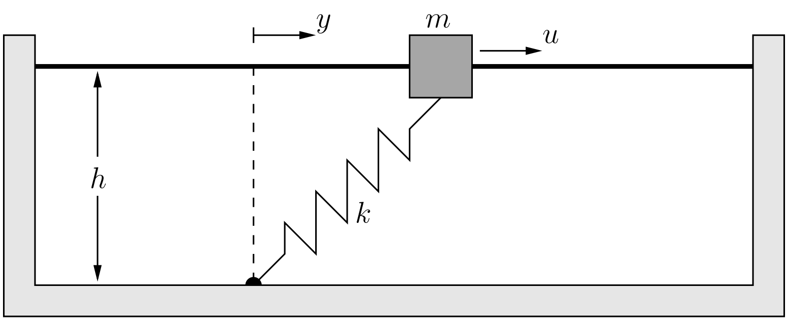
(c) $\ddot{y} + 3\dot{y}^2 + y^3 = u$

(d) $\left\{\begin{array}{lcl} \dot{y}_1 + 2(y_1 - y_2) & = & u_1 \\ \ddot{y}_2 + 4\dot{y}_2 + 3y_2 & = & u_2 \end{array}\right\}$

(e) $G(s) = \dfrac{s^2 + 3s + 1}{s^3 + 4s^2 + 2}$

(f) $\ddot{y} - 2\dot{y} - y = t\dot{u} + (t+1)u$

Q2. A mass m slides over a frictionless rigid bar and is connected to a spring with spring constant k. The spring is in an unstretched position when it is at an angle of 45 degrees from the vertical. The horizontal displacement of the mass from the vertical is y. The mass is subjected to a force input u.



(a) Obtain a state space description for this system.

(b) Find *all* equilibrium points of this system.

(c) Linearize the system around these equilibria.

Q3. Given two nonzero vectors $v,\, w \in \mathbb{R}^n$ with $v^T w \neq 0$, let $A = vw^T$.

(a) Find $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\operatorname{rank} A$.

(b) Find the eigenvectors, eigenvalues, characteristic polynomial, minimal polynomial, and Jordan form of A.

(c) Repeat part (b) for the case when $v^T w = 0$.

Q4. Find systems that, with zero input, can have the following outputs: **(a)** $y = t$, **(b)** $y = t^2$, **(c)** $y = t + t^2$, **(d)** $y = \cos t$, **(e)** $y = t \sin t$, **(f)** $y = t \sin t + \cos t$.

Q5. If λ is an eigenvalue of the matrix A, what is an eigenvalue of: **(a)** $-A$, **(b)** A^T, **(c)** A^{-1} (if it exists), **(d)** $2A$, **(e)** A^2, **(f)** e^A?

Q6. For an unknown matrix $A \in \mathbb{R}^{n\times n}$ suppose that you know its n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and the corresponding eigenvectors $v_1, v_2, \ldots, v_n$. Give explicit expressions for A and e^{At}.

Q7. For each of the following A matrices determine the eigenvalues, eigenvectors, and e^{At}.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Q8. Suppose that your software can solve only the systems without inputs: $\dot{x} = Ax$. Can you still use it to solve $\dot{x} = Ax + bu$ when $u(t) = e^{-t}$? How? What if $u(t) = \frac{1}{t+1}$?

Q9. For which $a, b \in \mathbb{R}$ will the below matrix be e^{At} of some A? *Hint: You may want to use the properties of matrix exponential.*

$$\begin{bmatrix} (1+at)e^{at} & -4bte^{at} \\ bte^{-2bt} & (1-at)e^{-2bt} \end{bmatrix}$$

Q10. Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ let

$$\mathcal{C} = [B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B] \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

and for $x \in \mathbb{R}^n$ prove the following.

(a) $x \in \mathcal{N}(\mathcal{O}) \implies Ax \in \mathcal{N}(\mathcal{O})$.

(b) $x \in \mathcal{R}(\mathcal{C}) \implies Ax \in \mathcal{R}(\mathcal{C})$.

Q11. For $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ show that $e^{At} = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$

Q12. Consider $A \in \mathbb{R}^{n\times n}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\ell_i \in \mathbb{C}^n$ be the ith *left eigenvector* of A, that is, $\ell_i^T A = \lambda_i \ell_i^T$. Let $r_i \in \mathbb{C}^n$ denote the ith (right) eigenvector. Since ℓ_i cannot be orthogonal to r_i (verify this!) we can and do choose our eigenvectors to satisfy $\ell_i^T r_i = 1$ and then define $A_i := r_i \ell_i^T$. Now prove the following.

(a) $A_1 + A_2 + \ldots + A_n = I$.

(b) $\lambda_1 A_1 + \lambda_2 A_2 + \ldots + \lambda_n A_n = A$.

(c) $e^{\lambda_1 t} A_1 + e^{\lambda_2 t} A_2 + \ldots + e^{\lambda_n t} A_n = e^{At}$.

(d) $AA_i = A_iA = \lambda_i A_i$.

(e) $A_iA_j = 0$ for $i \neq j$.

Q13. Consider

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x\,, \quad y = [1\ \ 1\ \ 1]x\,.$$

Choose, if possible, an initial condition $x(0)$ such that $y(t) = te^{-t}$ for $t \geq 0$.

Q14. Consider

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u\,, \quad y = [1\ \ 0]x\,.$$

Determine $x(0)$ so that for $u(t) = e^{-4t}$ we have $y(t) = ke^{-4t}$ where $k \in \mathbb{R}$. Determine k.

Q15. Determine the state transition matrix $\Phi(t,\,\tau)$ for the system $\dot{x} = \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix} x$. Find the solution $x(t)$ for $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Q16. For a second order LTV system two solutions are known to be

$$x_a(t) = \begin{bmatrix} 1/t^2 \\ -1/t \end{bmatrix} \quad \text{and} \quad x_b(t) = \begin{bmatrix} 2/t^3 \\ -1/t^2 \end{bmatrix}$$

(a) Find the solution $x(t)$ for $x(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. (Do not compute the state transition matrix here!)

(b) Verify that the solutions $x_a(t)$ and $x_b(t)$ may belong to the below system.

$$\dot{x} = \begin{bmatrix} -4/t & -2/t^2 \\ 1 & 0 \end{bmatrix} x$$

(c) Determine the state transition matrix $\Phi(t,\,\tau)$.

(d) Verify your answer to part (a) using $\Phi(t,\,\tau)$.

Q17. Show that the two linear systems

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 2-t^2 & 2t \end{bmatrix} x \quad \text{and} \quad \dot{z} = \begin{bmatrix} t & 1 \\ 1 & t \end{bmatrix} z$$

are equivalent state space representations of the differential equation $\ddot{y} - 2t\dot{y} - (2-t^2)y = 0$.

Q18. Consider the systems $\dot{x} = A(t)x$ and $\dot{z} = -A(t)^T z$ with the state transition matrices $\Phi(t,\,\tau)$ and $\Psi(t,\,\tau)$, respectively.

(a) Show that $\dot{\Phi}(\tau,\,t) = -\Phi(\tau,\,t)A(t)$. *Hint: differentiate the equation* $\Phi(\tau,\,t)\Phi(t,\,\tau) = I$.

(b) Show that $\Psi(t,\,\tau) = \Phi(\tau,\,t)^T$.

(c) Show that $\langle x(t),\, z(t)\rangle = \langle x(t_0),\, z(t_0)\rangle$ for all $t \in \mathbb{R}$.

Q19. Given $e^{At} = \begin{bmatrix} 1-2t & 2t & 2t \\ \frac{1}{2}(1-4t-e^t) & \frac{1}{2}(1+4t+e^t) & \frac{1}{2}(1+4t-e^t) \\ \frac{1}{2}(-1+e^t) & \frac{1}{2}(1-e^t) & \frac{1}{2}(1+e^t) \end{bmatrix}$

(a) Determine A.

(b) Determine the Jordan form of A.

Q20. Consider the following two systems.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad & \dot{z} &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y &= [1 \ \ -1 \ \ 0]x & y &= [1 \ \ -1 \ \ 0]z \end{aligned}$$

(a) Are these systems zero-state equivalent?

(b) Are they algebraically equivalent?