Q1. Consider the system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$. Suppose the following inequality holds

$$A^T P + PA + 2\mu P < 0$$

for some symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive real number μ. Show that all the eigenvalues λ_i of A must satisfy $\mathrm{Re}\{\lambda_i\} < -\mu$. (When the matrix A is as such, the system $\dot{x} = Ax$ is said to be asymptotically stable with *stability margin* μ.) *Hint: Try first to rewrite the inequality as* $[\text{something}]^T P + P[\text{something}] < 0$.

Q2. Consider the system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$. For each of the below cases determine the stability property of the system (exponentially stable, stable, or unstable).

(a) $A^T + A < 0$.

(b) $A^T + A = 0$.

(c) $A^T + A > 0$.

Q3. Consider the system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$. Suppose $A^T + A = 0$.

(a) Show that the solutions satisfy $x(t)^T x(t) = x(0)^T x(0)$ for all t.

(b) Show that the state transition matrix e^{At} is orthogonal for all t. (Recall that a square matrix Q is orthogonal if $Q^T Q = I$.)

(c) Show that A cannot have any 2×2 or larger Jordan block. *Hint: First show that all the eigenvalues of A must reside on the imaginary axis.*

Q4. Consider the system $x^+ = Ax$ with $x \in \mathbb{R}^n$. Suppose the matrix A satisfies

$$A^T PA - \rho^2 P < 0$$

for some symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive real number ρ. Show that all the eigenvalues λ_i of A must satisfy $|\lambda_i| < \rho$. Show also that we can find a positive number c such that any solution satisfies $\|x(k)\| \le c\rho^k \|x(0)\|$ for $k = 0, 1, 2, \ldots$

Q5. Consider the system $\dot{x} = PAx$ with $P, A \in \mathbb{R}^{n\times n}$. Suppose $A^T + A < 0$ and $P^T = P > 0$. Show that this system is exponentially stable.

Q6. Given $A \in \mathbb{R}^{n\times n}$ let the Lyapunov equation

$$A^T P + PA = -Q$$

hold for symmetric positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$. Using the eigenvectors of A show that this equation implies that the eigenvalues of A must be with negative real parts.

Q7. Consider the system $x^+ = Ax$ with $x \in \mathbb{R}^n$. Suppose for any initial condition $x(0) = x_0$ the solution satisfies $x(k) = 0$ for some finite k. What can be said about the eigenvalues of A?

Q8. Consider the following second order LTV system

$$\dot{x} = A(t)x \quad \text{where} \quad A(t) = -\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} [\cos t \;\; \sin t] = -\begin{bmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{bmatrix}$$

(a) Using $v(x) := x^T x$ as a Lyapunov function show that this system is stable.

(b) Now we will show that this system exponentially stable via change of variables. Define the new state $z(t) := Q(t)x(t)$ where

$$Q(t) := \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Prove the following: (i) $\|x(t)\| = \|z(t)\|$ for all t. (In particular, $Q(t)^T Q(t) = I$.) (ii) The solution $z(t)$ is described by an LTI system. (iii) The LTV system is exponentially stable. *Hint: Observe that* $\dot{Q} = Q \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} Q$.

(c) Propose a time-varying Lyapunov function of the form $w(x, t) = x^T P(t) x$ where $P(t)^T = P(t)$ for all t satisfies $c_1 I \le P(t) \le c_2 I$ and $\dot{w} \le -c_3 w$ for some fixed c_1, c_2, $c_3 > 0$.

Q9. Consider the system $x^+ = Ax$ with $x \in \mathbb{R}^n$. Suppose that this system is exponentially stable. Given $Q^T = Q > 0$ show that the matrix P defined as

$$P := Q + A^T Q A + A^{2T} Q A^2 + A^{3T} Q A^3 + \dots$$

is well-defined (i.e., the infinite sum converges), symmetric positive definite, and solves the Stein equation $A^T P A - P = -Q$.

Q10. Consider the system $x^+ = e^A x$ with $x \in \mathbb{R}^n$. Given that this discrete-time system is exponentially stable, show that the continuous-time system $\dot{x} = Ax$ must also be exponentially stable.

Q11. Determine the stability of the below systems

$$\dot{x} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x\,, \qquad \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} x\,.$$

Q12. Determine the stability of the below systems

$$x^+ = \begin{bmatrix} e^{-1} & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x\,, \qquad x^+ = \begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x\,.$$

Q13. Determine the stability of the below systems

$$x^+ = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} x\,, \qquad \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x\,.$$

Q14. Determine the (local) stability of the origin of the below systems

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x}_1 &= -x_1 + x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_2 + x_2(x_1^2 + x_2^2) \end{aligned}$$

by linearization if possible. Otherwise try simulation.

Q15. Determine the (local) stability of each equilibrium point of the below system, known as either Lotka-Volterra equations or the predator-prey equations,

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_2) \\ \dot{x}_2 &= -x_2(1 - x_1) \end{aligned}$$

by linearization if possible. Otherwise try simulation.