Q1. Consider the frictionless mass spring system

$$m\ddot{y} + ky = u$$

where constants m and k are the mass and the spring constant, respectively. The output y is the position of the mass and the input u is the force applied to the system.

(a) Is this system stable in the sense of Lyapunov?

(b) Is this system BIBO stable? If not, suggest a bounded input that would result in an unbounded output.

Q2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -5 & 4 \\ -8 & 7 \end{bmatrix} x + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u \\ y &= [2 \ \ \beta] x \end{aligned}$$

(a) Is this system stable in the sense of Lyapunov?

(b) Find α so that the system is BIBO stable for all β.

(c) Find β so that the system is BIBO stable for all α.

Q3. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

with nonzero matrices $B \in \mathbb{R}^{2\times 1}$ and $C \in \mathbb{R}^{1\times 2}$. This system is known to be unstable in the sense of Lyapunov. Show that if the system is BIBO stable then $A \in \mathbb{R}^{2\times 2}$ cannot have complex eigenvalues.

Q4. Show that an LTV system $\dot{x} = A(t)x + B(t)u$ is controllable on some interval $[t_0,\, t_1]$ if and only if it is reachable on $[t_0,\, t_1]$.

Q5. Let $\mathcal{R}$ and $\mathcal{C}$ respectively denote the reachable and controllable subspaces of an LTV system. Given two times $t_1 > t_0$ and a positive number $\varepsilon > 0$ determine whether each of the below claims is true or false.

(a) $\mathcal{R}[t_0,\, t_1] \subset \mathcal{R}[t_0,\, t_1 + \varepsilon]$.

(b) $\mathcal{R}[t_0,\, t_1] \subset \mathcal{R}[t_0 - \varepsilon,\, t_1]$.

(c) $\mathcal{C}[t_0,\, t_1] \subset \mathcal{C}[t_0,\, t_1 + \varepsilon]$.

(d) $\mathcal{C}[t_0,\, t_1] \subset \mathcal{C}[t_0 - \varepsilon,\, t_1]$.

Q6. Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times k}$ prove or disprove (by constructing a counterexample) the below claims.

(a) If the pair $(A,\, B)$ is controllable then so is the pair $(e^A,\, B)$.

(b) If the pair $(e^A,\, B)$ is controllable then so is the pair $(A,\, B)$.

(c) If the pair $(A,\, B)$ is controllable then so is the pair $(A^{-1},\, B)$ for A nonsingular.

(d) If the pair $(A,\, B)$ is controllable then so is the pair $(A^T,\, B)$.

Q7. Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$, suppose both $(A,\, B)$ and $(A^T,\, B)$ pairs be controllable. Further assume that all the eigenvalues of A are distinct. Show that the below system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x \end{aligned}$$

is BIBO stable if and only if it is asymptotically stable. *Hint: See Q12(c) of HW1.*

Q8. Consider the state equation $\dot{x} = Ax + Bu$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times k}$, with $(A,\, B)$ controllable. Let the linear state-feedback control law be $u = Kx + Gv$, where $K \in \mathbb{R}^{k\times n}$ and $G \in \mathbb{R}^{k\times k}$, with G nonsingular. Show that the closed-loop system $\dot{x} = (A+BK)x + BGv$, where v is now our new input, is controllable. *What we are proving here is fundamental: Controllability is invariant under state feedback !*

Q9. Given a controllable pair $(A,\, B)$ suppose there exists $P = P^T > 0$ such that $A^T P + PA \leq 0$. Prove then that the matrix $[A - BB^T P]$ is exponentially stable.

Q10. Here we provide an application of Q9. Consider a pair of generalized harmonic oscillators

$$\begin{aligned} \dot{x}_1 &= Ax_1 + Bu_1 \\ y_1 &= B^T x_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x}_2 &= Ax_2 + Bu_2 \\ y_2 &= B^T x_2 \end{aligned}$$

where A is skew-symmetric and the pair $(A,\, B)$ is controllable. Suppose now that we couple these systems through their outputs by letting $u_1 = \alpha(y_2 - y_1)$ and $u_2 = \alpha(y_1 - y_2)$, where α is some positive scalar. Show that the coupled harmonic oscillators synchronize, that is, $\|x_1(t) - x_2(t)\| \to 0$ as $t \to \infty$. *Hint: write the differential equation for the error* $e := x_1 - x_2$.

Q11. Consider the system

$$\dot{x} = Jx + Bu$$

where the matrix J is block diagonal with each block having the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

with $\lambda_i \neq \lambda_j$ for $i \neq j$. Determine conditions on B that are necessary and sufficient for controllability.

Q12. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. Show that if A has two or more linearly independent eigenvectors that share the same eigenvalue then the pair $(A,\, B)$ is not controllable.

Q13. Consider the system in controllable canonical form

$$\dot{x} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

(a) Show that this system is controllable.

(b) Show that $\det(sI - A) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$.

(c) Find the feedback gain $K = [k_1 \;\; k_2 \;\; k_3]$ such that the feedback law $u = -Kx$ places the eigenvalues of the closed-loop system matrix $[A - BK]$ to $\{-1, \, -2 + j, \, -2 - j\}$.

Q14. For the pair $(A, \, B)$ the *controllability index* ρ is defined as the smallest nonnegative integer satisfying

$$\text{rank}\, [B \;\; AB \;\; \ldots \;\; A^{\rho-1}B] = \text{rank}\, [B \;\; AB \;\; \ldots \;\; A^{\rho}B]$$

Prove that for any $\ell \geq \rho$

$$\text{rank}\, [B \;\; AB \;\; \ldots \;\; A^{\rho-1}B] = \text{rank}\, [B \;\; AB \;\; \ldots \;\; A^{\ell}B]$$

Q15. Show that the pair $(A, \, B)$ is controllable if and only if the only $n \times n$ matrix X satisfying both $XA = AX$ and $XB = 0$ is $X = 0$. *Hint: use the right and left eigenvectors of A.*

Q16. Show that if the below system

$$\dot{x} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u$$

is controllable then so is the pair $(A_{22}, \, A_{21})$.