Q1. Consider the system

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\,.\end{aligned}$$

(a) Show that the system is observable if and only if

$$y(t) = 0 \quad \forall t \implies x(t) = 0 \quad \forall t\,.$$

(b) Show that the system is detectable if and only if

$$y(t) = 0 \quad \forall t \implies \lim_{t\to\infty} x(t) = 0\,.$$

Q2. Consider the system

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned}$$

with $A \in \mathbb{R}^{2\times 2}$ and $C \in \mathbb{R}^{1\times 2}$. Suppose for some initial condition $x(0)$ the output is measured to be $y(t) = \sin(t)$. Show that this system must be observable.

Q3. Find matrices $A \in \mathbb{R}^{3\times 3}$ and $C \in \mathbb{R}^{1\times 3}$ and an initial condition $x(0) \in \mathbb{R}^3$ such that the system

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned}$$

is unobservable yet detectable. In addition $y(t) = \sin(t)$.

Q4. **(a)** Find a sufficient condition on matrix A such that the following holds: Given any B, the pair (A, B) is controllable if and only if it is stabilizable.

(b) Find a sufficient condition on matrix A such that the following holds: For all B the pair (A, B) is stabilizable.

Q5. Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x \\ y &= [\alpha \;\; 2 \;\; 1]\, x\,.\end{aligned}$$

(a) Find all possible values for α such that the system is unobservable but detectable.

(b) For the α you found in part (a) compute the unobservable subspace.

(c) Find all possible values for α such that the system is not detectable.

Q6. Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} x \end{aligned}$$

This system is known to be unobservable. Find a nonzero $B \in \mathbb{R}^{3\times 1}$ such that the pair (A, B) is guaranteed to be uncontrollable. For the B you found, compute the transfer function matrix.

Q7. Find a minimal realization for the single input single output system whose impulse response is $h(t) = te^{-t}$. Find also a nonminimal realization that is both controllable and detectable.

Q8. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [1 \ \ -1]\, x\,. \end{aligned}$$

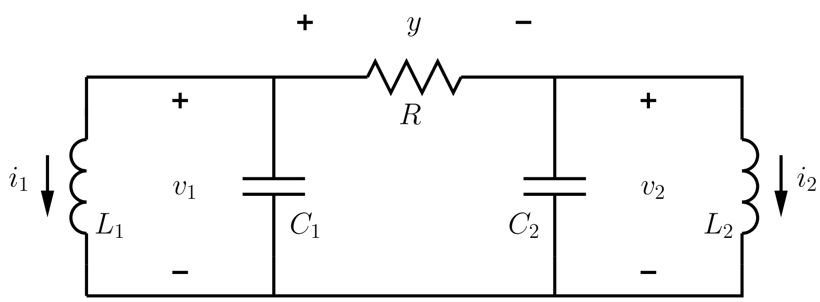
(a) Is this system stable, controllable, observable?

(b) Can this system be stabilized by static output feedback $u = -ky$?

(c) Stabilize the system via observer-based state feedback. In doing so, place the eigenvalues of the matrix $[A - LC]$ at $\lambda_{1,2} = -2, -3$ and the eigenvalues of the matrix $[A - BK]$ at $\lambda_{1,2} = -1 \pm j$. Also, write down the dynamics for the closed loop.

(d) Simulate the closed-loop system for various initial conditions $x(0)$ and $\hat{x}(0)$.

Q9. Consider the below circuit. Suppose $L_1C_1 = L_2C_2$. Show that this circuit is unobserv-



able from the output $y = v_1 - v_2$. How should one choose the (nonzero) initial conditions $[i_1(0),\, v_1(0),\, i_2(0),\, v_2(0)]$ such that $y(t) = 0$ for all t? *Hint: You may first want to show that the output satisfies a second-order differential equation of the form* $\ddot{y} + \alpha\dot{y} + \beta y = 0$.

Q10. Consider the system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

where $\mathbb{R}^{1\times 2} \ni C \neq 0$ and $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) Show that this system is observable.

(b) Now, consider the sampled-data version of this system

$$\begin{aligned} x^+ &= e^{AT}x \\ y &= Cx\,. \end{aligned}$$

Find (if possible) a condition on the sampling period $T > 0$ such that the discrete-time system is unobservable. [Remark: It turns out that the discrete-time system $x^+ = Fx$, $y = Kx$ is observable if and only if the continuous-time system $\dot{x} = Fx$, $y = Kx$ is observable.]

Q11. Consider the below single-input systems

$$\begin{aligned} \dot{x} &= Ax + B_1 u \\ y &= Cx \end{aligned} \qquad \text{and} \qquad \begin{aligned} \dot{x} &= Ax + B_2 u \\ y &= Cx \end{aligned}$$

Suppose that the pair $(C,\, A)$ is observable and the two systems have the same impulse response. Show that $B_1 = B_2$.

Q12. Show that the pair $(C,\, A)$ is observable if and only if the pair $(C^T C,\, A)$ is observable.

Q13. Consider the below n-dimensional ($n \geq 2$) single-input single-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

which is known to be both controllable and observable. Show that the matrices A and BC do not commute, i.e., $ABC \neq BCA$.

Q14. For what values of α the below system is the minimal realization of its transfer function?

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & \alpha & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \\ y &= [1 \;\; 0 \;\; 1]\, x\,. \end{aligned}$$

Q15. Consider the system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -4 & \alpha \\ \beta & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [-1 \;\; 2]\, x\,. \end{aligned}$$

It is known that this system is observable and has a bounded impulse response. Also, for some initial condition $x(0)$ and zero input $u = 0$, the output is measured to be $y(t) = 5e^{2t}$. Find α and β.