EE306: Signals and Systems II

Lecture 1

* About the course (3 modules).
* Applications (wireless communications, data storage, machine learning – internet of things).
* Importance of probability theory
  → What we are going to review.

Random experiment

It is an experiment in which the outcome varies in a non-deterministic manner when the experiment is repeated under the same conditions.

Ex 1 Roll a dice and note the number on the upper face of it (E_1).

Ex 2 Toss a coin 2 times and note the sequence (E_2).

Ex 3 Toss a coin 4 times and note # of tails (E_3).

Ex 4 Pick a real number at random bet. 0 and 1 (E_4).

Sample space

* Sample point : A result that cannot be decomposed into other results (outcome).
* Sample space S : The set of all possible outcomes.

Ex 1 $S(E_1) = \{1,2,3,4,5,6\}$.

Ex 2 $S(E_2) = \{HH, HT, TH, TT\}$.

Ex 3 $S(E_3) = \{0,1,2,3,4\}$.

$\left.\right\}$ Discrete S

Ex 4 $S(E_4) = \{x : 0 \leq x \leq 1\} = [0,1]$ (infinite).

Continuous S

Events

A situation specified by certain conditions, which occurs when the experiment outcome satisfies these conditions. Thus,

→ An event A corresponds to a subset of S.

→ The certain event is S.

The impossible event is ϕ.

Ex 1 An even number on upper face :

$A = \{2,4,6\}$.

Ex 2 At least one tail observed:

$$A = \{HT, TH, TT\}.$$

Ex 3 Five or more tails observed: $A = \phi$.

Ex 4 The number is such that $1 - x > 0.1$:

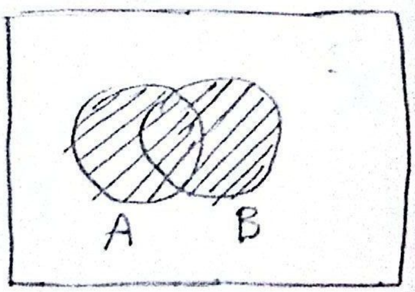
$$A = \{x : 0 \leq x < 0.9\} = [0, 0.9).$$

Probability: Number assigned to an event, indicating likelihood.

Basics of set theory

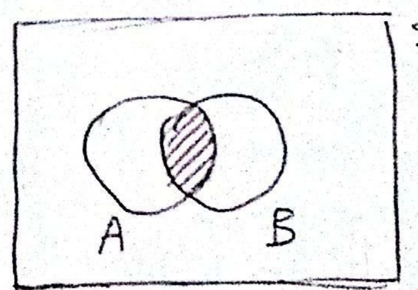
* Essential rotation using Venn diagrams:

$A \cup B$



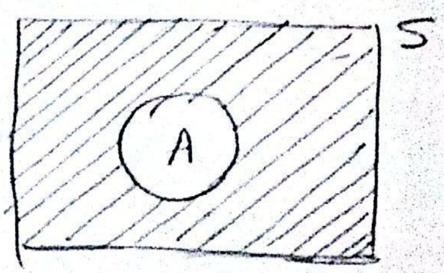
$\{x : x \in A \text{ or } x \in B\}$

$A \cap B$

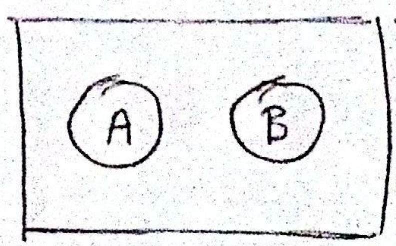


$\{x : x \in A \text{ and } x \in B\}$

$A^c = S - A$

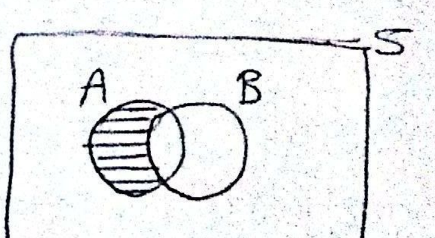


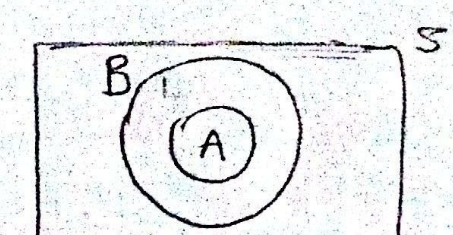
$\{x : x \notin A\}$



$A \cap B = \phi$ (disjoint)

$A - B = A \cap B^c$





$A \subset B$

$(A \subseteq B)$

Ex 1 An odd number on upper face :

$B = \{1,3,5\}$. Thus,

$A \cup B = \{1,2,\ldots,6\} = S$ and $A \cap B = \phi$.

Ex 2 Exactly one head observed :

$B = \{HT, TH\}$. Thus,

$A \cup B = \{HT, TH, TT\} = A$ because $B \subset A$.

Properties of set operations

* Commutativity : $A \cup B = B \cup A$, $A \cap B = B \cap A$.
* Associativity : $A \cup (B \cup C) = (A \cup B) \cup C$.

$A \cap (B \cap C) = (A \cap B) \cap C$.
* Distributivity : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Brackets are very important!

* DeMorgan's rules :

$(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

Ex 1 Apply DeMorgan to A and B :

$A^c = \{1,3,5\}$, $B^c = \{2,4,6\}$ $(A \cap B)^c = \phi^c = S = A^c \cup B^c$.

* Set equality: $A=B \iff A\subset B$ and $B\subset A$.

* $A\subset B$ means: B necessary for A
  while A sufficient for B.

* $\bigcup_{k=1}^{n} A_k = A_1\cup A_2\cup\cdots\cup A_n$, $\bigcap_{k=1}^{n} A_k = A_1\cap A_2\cap\cdots\cap A_n$.

<u>Probability axioms</u>

I. $P(A)\geq 0$. II $P(S)=1$.

III. If $A_1, A_2, \ldots, A_n$ are pairwise disjoint, i.e.,
$A_i\cap A_j=\phi$, $\forall\, i\neq j$, then $\underline{n>1}$ $P\left(\bigcup_{k=1}^{n} A_k\right)=\sum_{k=1}^{n} P(A_k)$.

<u>Corollary</u> $P(A^c)=1-P(A)$.

<u>Proof</u> $A\cap A^c=\phi$; thus, $P(A\cup A^c)=P(A)+P(A^c)$.
But $A\cup A^c=S$ and $P(S)=1 \Rightarrow P(A^c)=1-P(A)$. ▨
$\rightarrow$ (1)

<u>Corollary</u> $P(A)\leq 1$.

<u>Proof</u> From (1), $P(A)=1-P(A^c)$
Since $P(A^c)\geq 0$, $P(A)\leq 1$. ▨

<u>Corollary</u> $P(\phi)=0$. Easy to prove.

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Corollary $P(A\cup B) = P(A) + P(B) - P(A\cap B)$.

Proof We can write $A\cup B = \overbrace{(A\cap B^c)}^{E_1} \cup \overbrace{(B\cap A^c)}^{E_2} \cup \overbrace{(A\cap B)}^{E_3}$,

where E_1, E_2, and E_3 are pairwise disjoint.

Thus, $P(A\cup B) = P(E_1) + P(E_2) + P(E_3)$. $\rightarrow$ (2)

But $P(E_1) = P(A) - P(A\cap B)$ and $P(E_2) = P(B) - P(A\cap B)$.

Substituting in (2) proves the corollary. ▨

Corollary The inclusion-exclusion principle:

$$P\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\{i_1, i_2, \dots, i_k\} \\ \subseteq \{1,2,\dots,n\}}} P\left(\bigcap_{j=1}^{k} A_{i_j}\right).$$

Proof By induction using previous corollary (HW).

Corollary If $A \subseteq B$, then $P(A) \leq P(B)$.

Proof We can write $B = \overbrace{(B\cap A^c)}^{E_1} \cup \overbrace{(B\cap A)}^{E_2}$,

where E_1 and E_2 are disjoint.

Thus, $P(B) = P(B\cap A^c) + P(B\cap A)$.

But $B\cap A = A \Longrightarrow P(A) = P(B) - \underbrace{P(B\cap A^c)}_{\geq 0}$

$\Longrightarrow P(A) \leq P(B)$. ▨

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Lecture 2

Conditional probability

* For two events A and B :

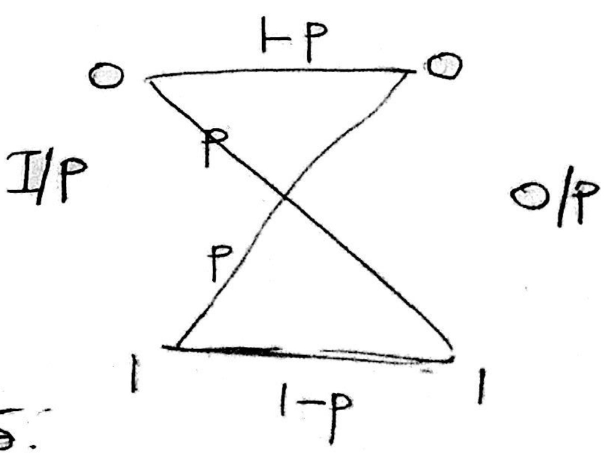
$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \text{ for } P(B)>0.$$

The relative frequency of the event $A\cap B$ in experiments where B occured, i.e., $n_{A\cap B}/n_B$.

* $P(A\cap B) = P(A|B)\,P(B) = P(B|A)\,P(A)$.

Ex 1 Binary symmtric channel

characterized as follows :



A_i : Event I/P is i.

B_j : Event O/P is j.

Error occurs with probability p, and $P(A_0) = 1-\delta$, $P(A_1) = \delta$.

Find $P(A_i \cap B_j)$, $\forall\, i,j \in \{0,1\}$.

Sol. $P(A_i\cap B_j) = P(B_j|A_i)\,P(A_i)$.

$P(B_j|A_i) = 1-p$, $i=j$, and $P(B_j|A_i) = p$, $i\neq j$.

Thus, $P(A_0, B_0) = P(B_0|A_0)P(A_0) = (1-p)(1-\delta)$.

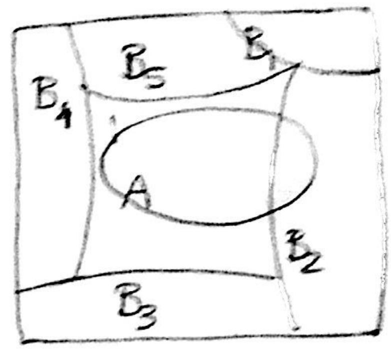
Similarly, $P(A_0, B_1) = p(1-\delta)$,

$P(A_1, B_0) = p\delta$, and $P(A_1, B_1) = (1-p)\delta$.

<u>Total probability theorem</u>

* Suppose $B_1, B_2, \ldots, B_n$ are mutually exclusive events such that $\bigcup_{k=1}^{n} B_k = S$.

B_k, $k \in \{1, 2, \ldots, n\}$ are a partition of S. Thus, $P(A) = \sum_{k=1}^{n} P(A|B_k)P(B_k)$.



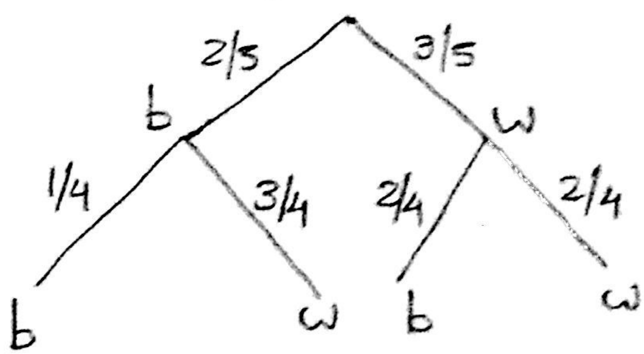
<u>Proof</u> $A = A \cap S = A \cap \left(\bigcup_{k=1}^{n} B_k\right)$

$= (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)$.

Thus, $P(A) = \sum_{k=1}^{n} P(A \cap B_k) = \sum_{k=1}^{n} P(A|B_k)P(B_k)$. ▨

<u>Ex2</u> An urn with 2 black and 3 white balls. Two balls are selected at random (no replacement) and the color is noted. Find the probability that second ball is black.

<u>Sol.</u> We build this tree:



$P(B_2) = P(B_2|B_1)P(B_1) +$
$P(B_2|W_1)P(W_1)$.

Thus, $P(B_2) = \left(\frac{1}{4}\right)\left(\frac{2}{5}\right) + \left(\frac{2}{4}\right)\left(\frac{3}{5}\right) = \frac{8}{20} = \frac{2}{5}$.

Bayes' rule

* We stick to the same partition of S. Suppose that event A occurs. Therefore,

$$P(B_j \mid A) = \frac{P(B_j \cap A)}{P(A)} = \frac{P(A \mid B_j)\, P(B_j)}{\sum_{k=1}^{n} P(A \mid B_k)\, P(B_k)}.$$

→ Very useful when

$P(A \mid B_k)$ and $P(B_k)$ are known for all k.

Ex 3 Posterior probabilities:

Consider the binary symmetric comm. channel of Ex 1.
Find $P(A_0 \mid B_0)$ and $P(A_1 \mid B_0)$. $\quad$ $P(tx \mid rx)$

Sol. $P(B_0) = P(B_0 \mid A_0)\, P(A_0) + P(B_0 \mid A_1)\, P(A_1)$

$= (1-p)(1-\delta) + p\delta$.

$$P(A_0 \mid B_0) = \frac{P(B_0 \mid A_0)\, P(A_0)}{P(B_0)} = \frac{(1-p)(1-\delta)}{(1-p)(1-\delta) + p\delta}.$$

$$P(A_1 \mid B_0) = \frac{P(B_0 \mid A_1)\, P(A_1)}{P(B_0)} = \frac{p\delta}{(1-p)(1-\delta) + p\delta}.$$

If $\delta = \frac{1}{2}$: $P(A_0 \mid B_0) = 1-p$, $P(A_1 \mid B_0) = p$.

Independence of events

* Events A and B are independent when:

$$P(A|B) = P(A) \quad \text{or} \quad P(A \cap B) = P(A)\,P(B).$$

* Events A, B, and C are independent when:

→ They are pairwise independent.

→ $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$.

$$\left[P(C|A \cap B) = \frac{P(A \cap B \cap C)}{P(A \cap B)} = P(C) \Rightarrow P(A \cap B \cap C) = P(A)\,P(B)\,P(C). \right]$$

Notes → $P(A,B|C) = P(A|B,C)\,P(B|C)$.

→ $P(A|C) = \sum_k P(A|B_k,C)\,P(B_k|C)$. $[, \Leftrightarrow \cap]$

Counting methods   n objects, k samples

* Sampling with replacement with ordering:

No. of options $= n^k$.

* Sampling without replacement with ordering:

No. of options $= {}^nP_k = \dfrac{n!}{(n-k)!}$.

* Sampling without replacement without ordering:

No. of options $= \dbinom{n}{k} = {}^nC_k = \dfrac{n!}{k!\,(n-k)!}$.

Idea: No. of permutations of k objects $= k!$, and they are all one option.

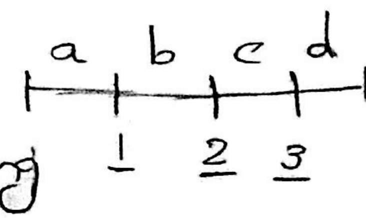
* Sampling with replacement without ordering:

$$\text{No. of options} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Idea: There are $k-1$ additional choices because repetition is allowed.

$$\text{* } P(\text{event}) = \frac{\text{No. of options satisfying the event}}{\text{Total no. of options}}.$$

Ex4: You want to partition a line with 4 segments into a number of parts between 2 and 4 by picking 3 of the specified cutting locations. Segments are labeled, and cutting locations can be repeated. How many arrangements?

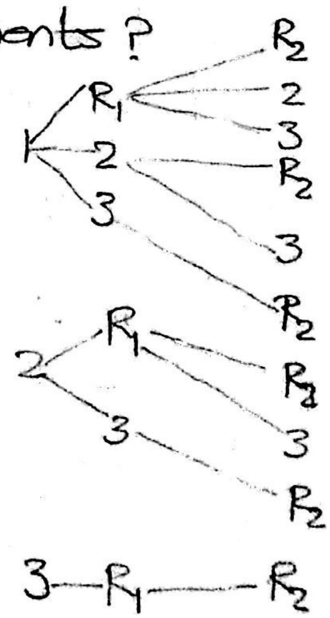


Sol. Here, $n=3$ and $k=3$.

We sample with replacement, no ordering. We get 2 new options:

$\{1, 2, 3, R_1, R_2\}$.

$$\text{Ans.} = \binom{n+k-1}{k} = \binom{5}{3} = 10.$$



Ex 5 There are n boxes labeled $1, 2, \dots, n$. There are also n balls labeled $1, 2, \dots, n$. Consider the case of $n \geq 2$. You place the balls into the boxes. Each ball has an equal probability of being placed at any box.

Find the probability that after placing all balls:

(a) Box 1 is (and possibly others are) empty.

(b) Exactly one box is empty.

Sol. No. of all possible arrangements $= n^n$.

(a) Probability is $\dfrac{(n-1)^n}{n^n}$.

(b) Probability is $\dfrac{n(n-1)\binom{n}{2}(n-2)!}{n^n} = \dfrac{\binom{n}{2}n!}{n^n}$.

[Choose empty box out of n —
choose box with 2 balls out of $n-1$ —
Choose 2 balls out of n for that box —
Arrange the rest a ball per box].