EE306: Signals and Systems II

Lecture 3

Random variables

* A random variable X is a function that assigns a real number $X(s)$ to each outcome s in the sample space of a random experiment.

→ Discrete random variables:
Probability mass function (PMF), $P_X(x)$.

→ Continuous random variables:
Probability density function (PDF), $f_X(x)$.

→ Cumulative distribution function (CDF), $F_X(x)$.

Expectation, variance, and moments

→ Discrete random variables:

$$E(X) = \sum_x x P_X(x) = m_X . \quad E(X^n) = \sum_x x^n P_X(x).$$

$$V(X) = E\left((X-m_X)^2\right) = E(X^2) - m_X^2 = \sigma_X^2 .$$

→ Important distributions:

Bernoulli: $P_X(0) = 1-p$, $P_X(1) = p$.

$$E(X) = p, \quad V(X) = p(1-p).$$

Binomial: Sum of independent Bernoulli's:

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \cdots, n.$$

$$E(X) = np, \quad V(X) = np(1-p).$$

Geometric: $P_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \cdots, \infty.$

$$E(X) = \frac{1}{p}, \quad V(X) = \frac{1-p}{p^2}.$$

EX1 Prove that if X is a geometric RV, $E(X) = \frac{1}{p}$.

Sol. $E(X) = \sum_x x P_X(x) = \sum_k k P_X(k)$. Thus,

$$E(X) = \sum_k k p (1-p)^{k-1} = p \sum_k k (1-p)^{k-1}$$

$$= p \sum_k \frac{d}{dp} \left[-(1-p)^k \right] = -p \frac{d}{dp} \left[\sum_k (1-p)^k \right]$$

$$= -p \frac{d}{dp} \left[\frac{1}{1-(1-p)} \right] = -p \left(\frac{-1}{p^2} \right) = \frac{1}{p}. \quad \blacksquare$$

Poisson: $P_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \cdots, \infty, \quad \lambda > 0.$

$$E(X) = \lambda, \quad V(X) = \lambda.$$

EX2 Let $p = \frac{\lambda}{n}$. Prove that a binomial RV with this p is approximately a Poisson RV for large n.

Sol. $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}.$

Now, for $p = \lambda/n$ and n large:

$$P_X(k) = \frac{\cancel{n}(\cancel{n-1})\cdots(\cancel{n-k+1})}{k!}\,\frac{\lambda^k}{\cancel{n^k}}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\approx \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^n \approx \frac{\lambda^k}{k!}e^{-\lambda}, \quad k=0,1,\ldots,\infty. \quad ▨$$

→ Continuous random variables:

$$F_X(x) = P(X \leq x), \quad P(a \leq X \leq b) = F_X(b) - F_X(a).$$

$$f_X(x) = \frac{dF_X(x)}{dx}, \quad F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt. \quad P(a \leq X \leq b) = \int_a^b f_X(x)\,dx.$$

→ Important distributions:

Uniform: $f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$

$$E(X) = \frac{a+b}{2}, \quad V(X) = \frac{(b-a)^2}{12}.$$

Exponential: $f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0, \lambda > 0.$

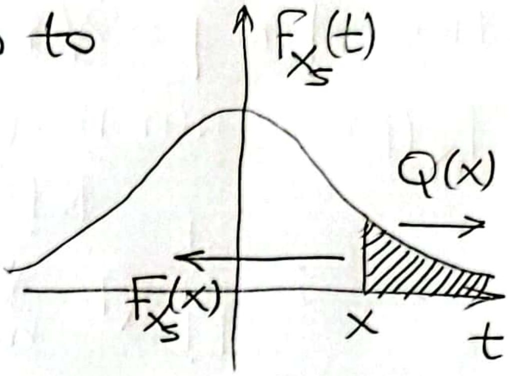
$$E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}.$$

* If the time between events is exponentially distributed with λ, the number of events per time unit is Poisson with $1/\lambda$.

Gaussian: $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-m_X)^2/2\sigma_X^2} \rightarrow N(m_X, \sigma_X^2)$

$$E(X) = m_X, \quad V(X) = \sigma_X^2.$$

* Finding probability via conversion to standard Gaussian $X_s \sim \mathcal{N}(0,1)$.



$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2}\, dt = 1 - F_{X_s}(x).$$

<u>Additional notes</u> $E(aX+b) = aE(X)+b$, $V(aX+b) = a^2 V(X)$.

$$E\Big(\sum_i X_i\Big) = \sum_i E(X_i), \quad V\Big(\sum_i X_i\Big) = \sum_i V(X_i) + \sum_{i \neq j} CV(X_i, X_j).$$

<u>Ex 3</u> You want to attend m football games. To do that, you need to collect m distinct tickets. You have access to infinitely many folded tickets, and tickets are equally likely. What is the expected number of folded tickets you need to unfold until you have at least one of each of the m tickets?

<u>Sol.</u> $E(T_1 + T_2 + \cdots + T_m) = E(T_1) + E(T_2) + \cdots + E(T_m)$.

But T_i is a geometric RV with $p = \frac{m-(i-1)}{m}$.

Thus, $E = \sum_i E(T_i) = \frac{1}{m/m} + \frac{1}{(m-1)/m} + \cdots + \frac{1}{1/m}$

$$= \sum_{i=1}^{m} \frac{m}{i} = m \sum_{i=1}^{m} \frac{1}{i}$$

$$= m H_m \approx \underline{m \ln m + O(m)}.$$