EE306 : Signals and Systems II

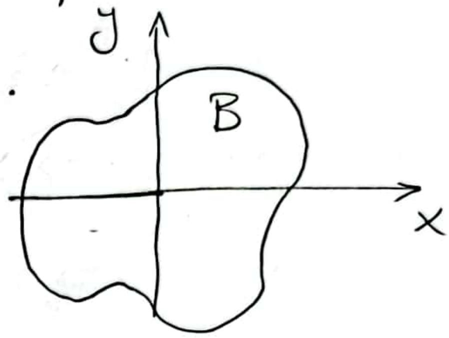
Lecture 6

* Pairs of random variables

→ Discrete RVs : Marginal PMFs

$$P_X(x) = \sum_y P_{X,Y}(x,y) \ , \ P_Y(y) = \sum_x P_{X,Y}(x,y).$$

$$P[(X,Y) \text{ in } B] = \sum\sum_{(x,y)\in B} P_{X,Y}(x,y).$$



(cont. X and Y)

→ Continuous RVs : Marginal PDFs

$$f_X(x) = \int_y f_{X,Y}(x,y)\, dy,$$

$$f_Y(y) = \int_x f_{X,Y}(x,y)\, dx.$$

$$P[(X,Y) \text{ in } B] = \iint_{\text{plane of } B} f_{X,Y}(x,y)\, dx\, dy.$$

→ CDF useful relations : Marginal CDFs

$$F_X(x) = F_{X,Y}(x,\infty) \ , \ F_Y(y) = F_{X,Y}(\infty, y).$$

$$F_{X,Y}(x,-\infty) = 0 \ , \ F_{X,Y}(-\infty, y) = 0.$$

$$P[x_1 < X \leq x_2 \ , \ y_1 < Y \leq y_2] = F_{X,Y}(x_2,y_2)$$
$$- F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1).$$

Ex1 A modem transmits a 2-D signal (X,Y) given by:

$$X = r\cos(2\pi\Theta/8) \ , \ Y = r\sin(2\pi\Theta/8).$$

where Θ is a discrete uniform RV in $S = \{0, 1, \ldots, 7\}$.

(a) Show the mapping from S to $S_{X,Y}$.

(b) Find $P_{X,Y}(x,y)$. (c) Find $P_X(x)$ and $P_Y(y)$.

(d) Find the prob. of the following events:

$A = \{X=0\}$, $B = \{Y \leq r/\sqrt{2}\}$, $C = \{X \geq r/\sqrt{2}, Y \geq r/\sqrt{2}\}$.

<u>Sol.</u> $S_{X,Y}$ is the joint sample space.

(a) $S(0) \rightarrow S_{X,Y}(r,0)$. $S(4) \rightarrow S_{X,Y}(-r,0)$.

$S(1) \rightarrow S_{X,Y}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$ $S(5) \rightarrow S_{X,Y}(\frac{-r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$.

$S(2) \rightarrow S_{X,Y}(0,r)$. $S(6) \rightarrow S_{X,Y}(0,-r)$.

$S(3) \rightarrow S_{X,Y}(\frac{-r}{\sqrt{2}}, \frac{r}{\sqrt{2}})$. $S(7) \rightarrow S_{X,Y}(\frac{r}{\sqrt{2}}, \frac{-r}{\sqrt{2}})$.

(b) $P_{X,Y}(x,y) = \frac{1}{8}$, $\forall x,y$ s.t. $(x,y) \in S_{X,Y}$.

(c) $S_X = \{-r, \frac{-r}{\sqrt{2}}, 0, \frac{r}{\sqrt{2}}, r\}$.

$$P_X(x) = \begin{cases} 1/4, & x \in \{\frac{-r}{\sqrt{2}}, 0, \frac{r}{\sqrt{2}}\} \\ 1/8, & x \in \{-r, r\} \end{cases}, \text{ and } 0 \text{ otherwise.}$$

The exact same PMF for Y as well.

(d) $P[A] = P_X(0) = \frac{1}{4}$.

$P[B] = 1 - P_Y(r) = 1 - \frac{1}{8} = \frac{7}{8}$.

$P[C] = P_{X,Y}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}) = \frac{1}{8}$.

<u>Ex2</u> The amplitudes of two signals X and Y have joint PDF: $f_{X,Y}(x,y) = e^{-x/2} y e^{-y^2}$, $x, y > 0$.

(a) Find the joint CDF.

(b) Find $P[\sqrt{X} > Y]$. (c) Find the marginal PDFs.

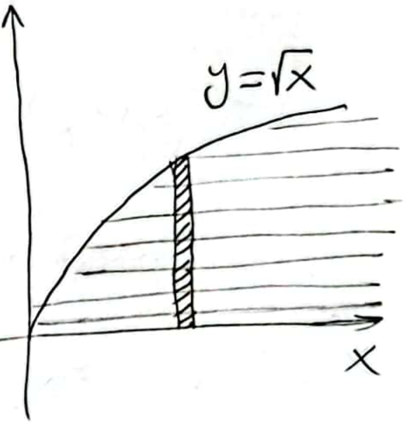
Sol. (a) $F_{X,Y}(x,y) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{y} f_{X,Y}(u,v)\,dv\,du \Rightarrow$

$$F_{X,Y}(x,y) = \int_{u=0}^{x}\int_{v=0}^{y} e^{-u/2}\, v e^{-v^2}\, dv\,du = \int_{u=0}^{x} e^{-u/2}\int_{v=0}^{y}\left(\frac{-1}{2}\right) d\left(e^{-v^2}\right) du$$

$$= \int_{u=0}^{x} e^{-u/2}\cdot\frac{1}{2}e^{-v^2}\Big|_{v=y}^{0} du = \frac{1}{2}\left(1-e^{-y^2}\right)(2)e^{-u/2}\Big|_{u=x}^{0}.$$

Thus, $F_{X,Y}(x,y) = \left(1-e^{-x/2}\right)\left(1-e^{-y^2}\right)$.

(b) $$P[\sqrt{X} > Y] = \int_{x=0}^{\infty}\int_{y=0}^{\sqrt{x}} e^{-x/2}\, y e^{-y^2}\, dy\,dx$$



$$= \int_{x=0}^{\infty} e^{-x/2}\int_{y=0}^{\sqrt{x}}\left(\frac{-1}{2}\right) d\left(e^{-y^2}\right) dx$$

$$= \int_{x=0}^{\infty} e^{-x/2}\cdot\frac{1}{2}\left(1-e^{-x}\right)dx = \int_{x=0}^{\infty}\frac{1}{2}\left(e^{-x/2}-e^{-3x/2}\right)dx$$

$$= \frac{1}{2}\left[2e^{-x/2}-\frac{2}{3}e^{-3x/2}\right]_{x=\infty}^{0} = \frac{1}{2}\left(2-\frac{2}{3}\right) = \frac{2}{3}.$$

(c) $F_X(x) = F_{X,Y}(x,\infty) = \left(1-e^{-x/2}\right)$.

Thus, $f_X(x) = \frac{dF_X(x)}{dx} = \frac{1}{2}e^{-x/2}$, $x>0$. (what is this?)

$F_Y(y) = F_{X,Y}(\infty,y) = \left(1-e^{-y^2}\right)$.

Thus, $f_Y(y) = \frac{dF_Y(y)}{dy} = 2ye^{-y^2}$, $y>0$.

Properties of CDFs and PDFs can be verified.

* Jointly Gaussian RVs

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-m_X}{\sigma_X}\right)^2 + \left(\frac{y-m_Y}{\sigma_Y^2}\right)^2 - 2\rho\left(\frac{x-m_X}{\sigma_X}\right)\left(\frac{y-m_Y}{\sigma_Y}\right)\right]\right]$$

Q What happens when $\rho = 0$?

* Conditional PMF/PDF & expectation

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}, \quad P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}.$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}.$$

→ $E(Y|X) = \int_y y f_{Y|X}(y|x)\,dy$ (sum if discrete).

→ Total expectation: $E_X[E(Y|X)] = E(Y)$.

Proof $E_X[E(Y|X)] = \int_x \int_y y f_{Y|X}(y|x) f_X(x)\,dy\,dx$.

Thus, $E_X[E(Y|X)] = \int_y y \int_x f_{Y|X}(y|x) f_X(x)\,dx\,dy$

$$= \int_y y \int_x f_{X,Y}(x,y)\,dx\,dy = \int_y y f_Y(y)\,dy.$$

Therefore, $E_X[E(Y|X)] = E(Y)$. ▨

→ $E[g(X)|X] = g(X)$.

Examples: $E[X|X] = X$ and $E[X^2|X] = X^2$.

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