EE306 : Signals and Systems II

Lecture 4

* Functions of random variables

→ CDFs and PDFs of functions of RVs $Y=g(X)$:

- Find the CDF

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$. If g^{-1} is unique:

$F_Y(y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$.

- Find $g^{-1}(y)$ and continue to get $F_Y(y)$.
- Differentiate the CDF to get the PDF.

→ You can also directly write (cont., what if discrete?)

$$f_Y(y) = \sum_k f_X(x_k) \left| \frac{dx}{dy} \right|_{x=x_k},$$

where each x_k is a possible $g^{-1}(y)$.

<u>Ex 1</u> Let the random variable Y be defined as $Y = X^2$, where X is a continuous RV with a known distribution. Find $F_Y(y)$ and $f_Y(y)$.

<u>Sol.</u> $F_Y(y) = P(Y \leq y) = P(X^2 \leq y)$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y \geq 0.$$

For $y<0$, $F_Y(y)=0$ since $Y = X^2 \geq 0$.

Next, $f_Y(y) = \dfrac{dF_Y(y)}{dy}$. Thus $f_Y(y) = 0$, $y \leq 0$,

and $f_Y(y) = \dfrac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \dfrac{1}{2\sqrt{y}} f_X(-\sqrt{y})$

$$= \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right).$$

→ Direct method: $\underbrace{Y = X^2 \Rightarrow y = x^2}_{Y = g(X) = X^2.}$. Thus,

$y = \left(g^{-1}(y)\right)^2 \Rightarrow g_1^{-1}(y) = x_1 = \sqrt{y}$

and $g_2^{-1}(y) = x_2 = -\sqrt{y}$.

Then, $f_Y(y) = \sum_{k=1}^{2} f_X(x_k) \left| \dfrac{dx}{dy} \right|_{x = x_k}$

$$= \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right).$$

<u>EX2</u> Computer generation of random variables.

→ Suppose you have RV U that is uniform in the interval $[0,1]$. Find $g(U)$ that gives X, an RV that is exponential with λ.

<u>Sol.</u> In general, let $U = h(X) = F_X(X)$.

Then, $F_U(u) = P(U \leq u) = P(F_X(X) \leq u)$

$$= P(X \leq F_X^{-1}(u)) = F_X(F_X^{-1}(u)).$$

This means $F_U(u) = u$, $0 \leq u \leq 1$.

Thus, $U = F_X(X)$ is a uniform RV in $[0,1]$

for any continuous RV X! (Meaning?)

Since $U = F_X(X) \Rightarrow X = F_X^{-1}(U) = g(U)$.

We know that $F_X(x) = 1 - e^{-\lambda x}$, $x \geq 0, \lambda > 0$.

Thus, $y = 1 - e^{-\lambda g(y)} \Rightarrow -\lambda g(y) = \ln(1-y)$.

$\Rightarrow g(y) = \frac{-1}{\lambda} \ln(1-y) = F_X^{-1}(y)$.

Finally, $X = \frac{-1}{\lambda} \ln(1-U) = F_X^{-1}(U)$.

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* Transform methods $\quad E(g(X)) = \int_x g(x) f_X(x)\, dx$ (cont.)

→ characteristic function

$$\phi_X(\omega) = E(e^{j\omega X}) = \int_{x=-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx \quad \text{(cont. RVs)}.$$

Generating moments $E(X^n) = \frac{1}{j^n} \frac{d^n}{d\omega^n} \phi_X(\omega) \Big|_{\omega=0}$.

Q How can we show this is true?

→ Probability generating function

$$G_X(z) = E(z^X) = \sum_{x=0}^{\infty} P_X(x) z^x \quad \text{(non-neg. discr. RVs)}.$$

Generating probabilities

$$P_X(x) = \frac{1}{x!} \frac{d^x}{dz^x} G_X(z) \Big|_{z=0}.$$

It can also be used to generate moments.

Ex3 Let $Y = aX + b$, where X is Gaussian with m, σ^2. Find the characteristic fn of Y.

Sol. $\phi_Y(\omega) = E(e^{j\omega Y}) = E(e^{j\omega(aX+b)})$

$$= e^{j\omega b} E(e^{j\omega a X}).$$

$$E(e^{j\omega a X}) = \int_{x=-\infty}^{\infty} e^{j\omega a x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m)^2/2\sigma^2} dx.$$

$$\text{Exponent} = \frac{-1}{2\sigma^2}\left[x^2 - 2mx + m^2 - 2\sigma^2 j\omega a x\right]$$

$$= \frac{-1}{2\sigma^2}\left[x - (m + j\sigma^2\omega a)\right]^2 + \frac{1}{2\cancel{\sigma^2}}\left[2jm\cancel{\sigma^2}\omega a - \sigma^{\cancel{4}2}\omega^2 a^2\right].$$

Thus,
$$E(e^{j\omega a X}) = e^{j\omega m a - \sigma^2\omega^2 a^2/2} \underbrace{\int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-[x-(m+j\sigma^2\omega a)]^2/2\sigma^2} dx}_{=1}$$

$$\Rightarrow E(e^{j\omega a X}) = e^{j\omega m a - \sigma^2\omega^2 a^2/2}.$$

Therefore, $\phi_Y(\omega) = e^{j\omega b}\, e^{j\omega m a - \sigma^2\omega^2 a^2/2}$

$$= e^{j\omega(b+ma) - \omega^2\sigma^2 a^2/2}.$$

Mean $\frac{d}{d\omega}\phi_Y(\omega) = \left[j(b+ma) - \omega\sigma^2 a^2\right] e^{j\omega(b+ma) - \omega^2\sigma^2 a^2/2}$

$$\Rightarrow \frac{1}{j}\frac{d}{d\omega}\phi_Y(\omega)\Big|_{\omega=0} = \frac{1}{j}\left[j(b+ma)\right](1) = \underline{b + ma}.$$

$E(Y) = b + ma$ as expected.

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Lecture 5

* Probability bounds

→ Markov bound : For X non-negative

$$P[X \geq a] \leq \frac{E(X)}{a}.$$

→ Chebyshev bound : $E(X) = m$, $V(X) = \sigma^2$

$$P[|X-m| \geq a] \leq \frac{\sigma^2}{a^2} \text{ (better bound)}.$$

→ Chernoff bound :

$$P[X \geq a] \leq e^{-sa} E(e^{sX}).$$

Q How can we prove Markov bound?

Ex1 Compare the Markov bound and the exact probability for the event $\{X > c\}$ as a function of c for X being a binomial RV with $p=0.5$, $n=10$.

Sol. Markov bound : $P[X \geq c] \leq \frac{E(X)}{c} = \frac{np}{c}$.

$$P[X > c] = P[X \geq c] - P[X = c]$$

$$\leq \frac{np}{c} - \binom{n}{c} p^c (1-p)^{n-c}.$$

Exact prob. : $P[X > c] = 1 - P[X \leq c]$

$$= 1 - \sum_{k=0}^{c} \binom{n}{k} p^k (1-p)^{n-k}.$$

→ Bound vs exact for $p=0.5$:

$$\frac{n}{2c} - \binom{n}{c}(0.5)^n \geq 1 - \sum_{k=0}^{c} \binom{n}{k}(0.5)^n$$

$$\Rightarrow \frac{n}{2c} \geq 1 - \sum_{k=0}^{c-1} \binom{n}{k}(0.5)^n.$$

Now set $n=10$: $c=1$ LHS $=5$, RHS = some prob.

$c=10$ LHS $=0.5$, RHS $=(0.5)^{10}$.

Q What is your comment about the bound?

Ex 2 Let X be the number of successes in n Bernoulli trials, where the prob. of success is p. Let $Y = X/n$ be the average no. of successes per trial. Apply Chebyshev bound to the event $\{|Y-p|>a\}$. What happens as $n \to \infty$?

Sol. Chebyshev bound: $P[|Y-m| \geq a] \leq \frac{\sigma^2}{a^2}$.

$$m = m_Y = E(X/n) = \frac{1}{n}E(X) = \frac{1}{n}(np) = p$$

since X is a binomial RV.

$$\sigma^2 = \sigma_Y^2 = \frac{1}{n^2}V(X) = \frac{1}{n^2}(np(1-p)) = \frac{p(1-p)}{n}.$$

Thus, $P[|Y-p|>a] = P[|Y-m|>a]$

$$\leq P[|Y-m| \geq a] \leq \frac{\sigma^2}{a^2} = \frac{p(1-p)}{na^2}.$$

As $n \to \infty$, $P[|Y-p|>a] \to 0 \Rightarrow$ useful bound.

* Chebyshev bound relies on mean and variance.

Since a is arbitrary, y converges to $m=p$ as $n\to\infty$.

* Recursion

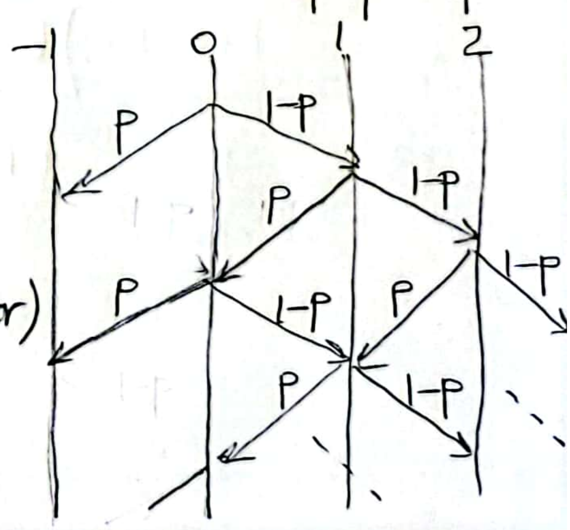
→ Find probability via $p=f(p)$.

→ Find expectation via $E=f(E)$.

→ Avoid searching for underlying distributions.

Ex 3 A man is playing chess games vs an online program at some difficulty level. The man loses a game with prob. p, and wins a game with prob. $1-p$, where p is fixed. He starts with a score of 0. His score increases (decreases) by 1 if he wins (loses) a game. As long as his score is non-neg., he keeps playing indefinitely. Once his score drops below 0, the online program blocks him.

→ Find the prob. the man is blocked if $p=\frac{1}{4}$.



Sol. Let $P(\text{blocked})=P_b$.

$P_b = P(\text{loses first game}) + P(\text{wins it but be blocked later})$

$\Rightarrow P_b = p + (1-p)\, P_{1\to 0}\, P_{0\to -1}$.

But $P_{1\to 0} = P_{0\to -1} = P_b$. Therefore,

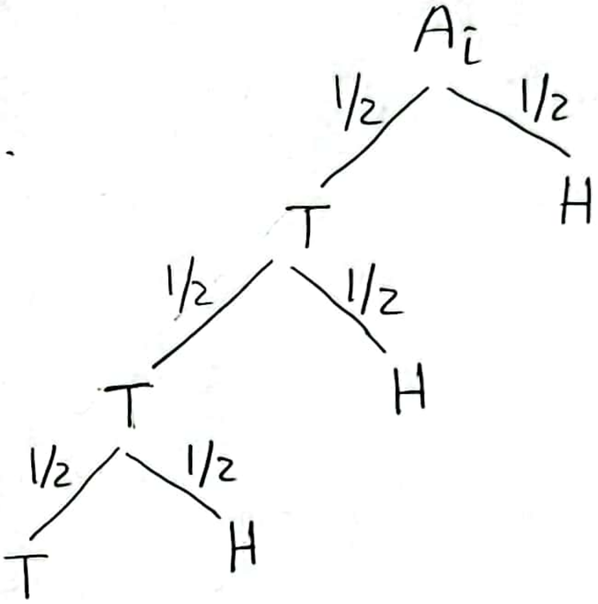
$$P_b = p + (1-p)P_b^2 \Rightarrow P_b = \frac{1 \pm \sqrt{1-4(1-p)p}}{2(1-p)}$$

$$\Rightarrow P_b = \frac{1 \pm \sqrt{1-4p+4p^2}}{2(1-p)} = \frac{1 \pm (1-2p)}{2(1-p)}$$

$$\Rightarrow \underbrace{P_b = 1}_{\text{for } p > \frac{1}{2}} \text{ or } \underbrace{P_b = \frac{p}{1-p}}_{\text{for } p \leqslant \frac{1}{2}} \Rightarrow P_b = \frac{1/4}{3/4} = \underline{\frac{1}{3}}.$$

<u>Ex 4</u> Find the expected number of fair coin tosses needed to get three successive tails.

<u>Sol.</u> $E(TTT) = E = \sum_i E(TTT \mid A_i)\, P(A_i)$.



Thus, $E = (E+1)\left(\frac{1}{2}\right) + (E+2)\left(\frac{1}{4}\right) + (E+3)\left(\frac{1}{8}\right) + (3)\left(\frac{1}{8}\right) \Rightarrow$

$$E = \left(\frac{4+2+1}{8}\right)E + \left(\frac{4+4+3+3}{8}\right)$$

$$\Rightarrow \frac{1}{8}E = \frac{14}{8}. \text{ Thus, } E = \underline{14}.$$