1.2.4 Steady-state probabilities

We observe from the three examples above, that when all recurrent classes in the MC are aperiodic, then as $n \to \infty$ the entries of the n-step transition probability matrix converge:

$$P_{ij}^{(n)} \to \pi_{ij}$$

Moreover, if the MC is also irreducible (that is, there is a single recurrent class, and possibly some transient states), these limiting values do not depend on the initial state:

$$\pi_{ij} = \pi_j$$

A Markov Chain which has a single recurrent class, that is also aperiodic, is *ergodic*. In this case, the limiting values $\{\pi_j\}$ are steady-state probabilities of observing each state j. Moreover, the value π_j is also equal to the long term fraction of time that state j is visited, in each sample path.

Ex: Consider two LEDs controlled by two switches in the following way: Switch 1 toggles the state of one of the LEDs, chosen equally likely at random (For example, if both LEDs are OFF (00), one of them will turn ON (01 or 10) when Switch 1 is flipped; from 10 or 01, if Switch 1 is flipped, they may go to 11 or 00.) Switch 2 turns both LEDs OFF.

Each minute, someone comes and flips one of the switches at random (each switch is chosen with equal probability.)

(a) Let the state of the system be the number of LEDs that are ON. Draw the diagram of a Markov Chain that models this process, and mark the transition probabilities.

(b) The P^n matrix converges to a matrix π in this problem. The entries of π depend only on the column number, not the row. That is, the probability of being in state j after a long amount of time is a constant number, that does not depend on the starting state i. In other words, the ij^{th} entry depends only on i, $\pi_{ij} = \pi_j$. (Note

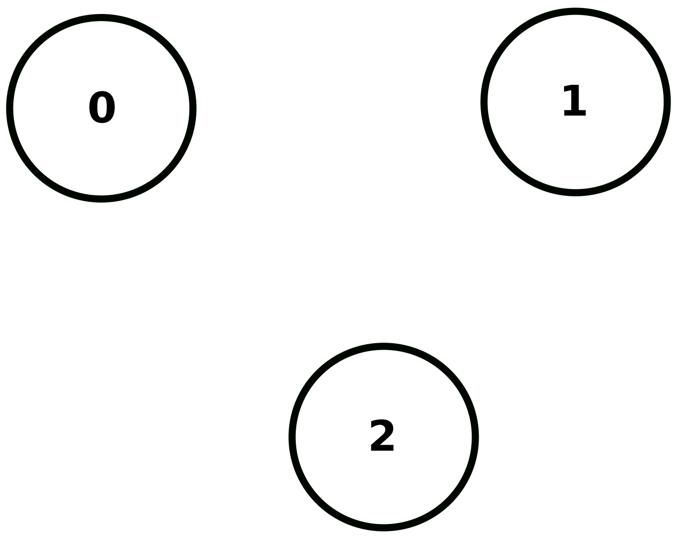

Figure 1.3: Ergodic Markov Chain for the switch flipping example: Mark the transition probabilities according to the probabilities described in the problem statement. All states are *recurrent*.

that $\pi_{ij} = lim_{n\to\infty} P(X_n = j|X_1 = i)$.) To find these limiting probabilities, we will apply the following logic: First, apply the Total Probability Theorem:

$$P(X_n = j) = \sum_{k=1}^{3} P(X_n = j|X_{n-1} = k)P(X_{n-1} = k)$$

If n is very large (large enough that steady state has been reached), than the probability of being in state k at time n should be the same as the probability of being in state k at time $n-1$. So, replace $P(X_{n-1} = k)$ by π_k and $P(X_n = k)$ also by π_k, for $k = 1, 2, 3$. We get a set of equations for the π_js:

$$\pi_j = \sum_{k=1}^{3} \pi_k p_{kj}$$

$j = 1, 2, 3$. In addition to these three equations (only two of which are linearly independent) we have another equation:

$$\sum_{j=1}^{3} \pi_j = 1$$

Solve this system of equations to find the steady-state probabilities.

(c) Suppose, after the switches have been flipped randomly as described above many times, I come in at a certain time. What is the probability that I see both LEDs ON? What is the long term fraction of time that both LEDs are ON?

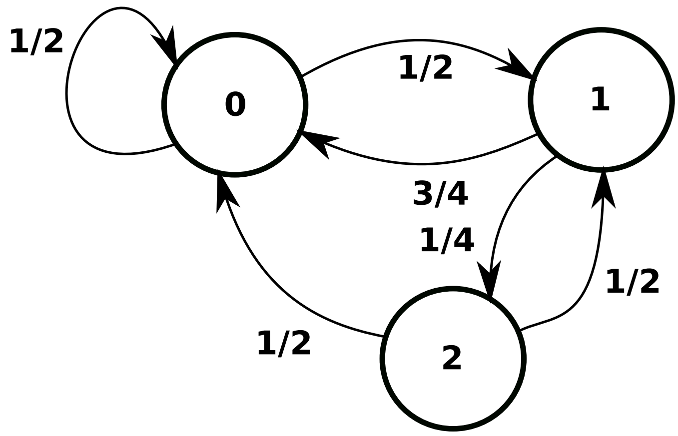

Figure 1.4: Ergodic Markov Chain for the switch flipping example with the transition probabilities marked. All states are *recurrent*.

Let us summarize these observations in a theorem that we will state without proof.

Theorem 2 *Consider a Markov Chain with $k < \infty$ states with a single recurrent class, which is aperiodic. Then, there is a set of values $0 \leq \pi_j \leq 1$, such that:*

[a]For each j,

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j, \forall i$$

The π_j are the unique solution satisfying $\sum_{m=1}^{k} \pi_m = 1$ to the set of equations:

$$\pi_j = \sum_{m=1}^{k} \pi_m P_{mj}, j = 1, 2, \ldots, kS \quad (1.1)$$

We can write the RHS in the above as a matrix multiplication:

$$\boldsymbol{\pi} = \boldsymbol{\pi} P$$

where $\boldsymbol{\pi}$ is the probability vector $[\pi_1 \pi_2 \ldots \pi_k]$.

Remarks:

- The set of **global balance equations** given in (1.1) contains only $m-1$ linearly independent equations (one equation will be redundant.) Therefore, (1.1) has an infinite number of solutions. We are looking for the unique solution that is a probability vector (i.e. satisfying the normalization $\sum_{m=1}^{k} \pi_m = 1$.)
- . We have

$$\pi_j = 0, \text{for all transient states } j \pi_j > 0, \text{for all recurrent states } j$$

- The π_j are called **stationary probabilities** because if the process started with these probabilities:

$$\mathbf{P}(X_o = j) = \pi_j, \forall j$$

  Then, by the Chapman-Kolmogorov eqns,

$$\mathbf{P}(X_1 = j) = \sum_m \mathbf{P}(X_o = m) P_{mj} = \sum_m \pi_m P_{mj} = \pi_j, \forall j$$

  Hence at any future time the distribution of the state is the same.

Long Term Frequency of Occurrence

Suppose, every day we visit state j, we get a unit reward. Given that we start in state i, what is the long term average reward per transition?

$$lim_{n\to\infty} \frac{r_{ij}(n)}{n}$$

In an ergodic process, the answer is equal to the expected reward per transition (We have to leave the proof of this intuitive result outside the scope in this course.)

$$lim_{n\to\infty} \frac{r_{ij}(n)}{n} = \pi_j$$

How about the long term frequency of transitions from j to k?

Expected frequency of a particular transition.

Balance equations:

Birth-death Chains:

Ex: Bank teller.

Mean First Passage and Recurrence Times